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Research article

Application of the More Generalized FBI Transform

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Abstract

This article explores the utility of the generalized class of Fourier–Bros– Iagolnitzer (FBI) transforms in problem-solving by presenting a specific problem that can be effectively addressed using the FBI transform with powers of 8, but not with lower powers.

Keywords: FBI transform, Gevrey function, Wave front Set.

1 Introduction

The Fourier transform is a powerful mathematical tool that has become fundamental in various fields of science and engineering. Introduced by Joseph Fourier in the early nineteenth century, the Fourier transform decomposes a function into its constituent frequencies, providing a frequency domain representation of the original signal.
Recall that for $u \in C_c^0(\mathbb{R}^m)$ the Fourier transform is defined by

$$\hat{u}(\xi) = \int_{\mathbb{R}^m} e^{-i\xi.x} u(x)dx, \ \xi \in \mathbb{R}^m$$

where $i = \sqrt{-1}$, $\xi.x = \sum_{j=1}^{m} x_j\xi_j$, when $u \in \mathcal{E}'(\mathbb{R}^m)$ the integral is understood in duality sense
This transforms are useful in the analysis of functions (distributions) and differential equations. For instance, the Paley-Wiener theorem states that: u is C^∞ on $\mathbb{R}^m$ if and only if for every $k = 1, 2, \ldots$, there exist $C_k > 0$ such that

$$|\hat{u}(\xi)| \leq \frac{C_k}{(1+|\xi|)^k}, \ \ \forall \xi \in \mathbb{R}^m.$$

The classical and more commonly used FBI (Fourier-Bros-Iagolnitzer) transform has the form

$$\mathcal{F}_u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)-|\xi||x-y|^2} u(y)dy,\ x,\xi \in \mathbb{R}^m, \tag{1.1}$$

where u is continuous function of compact support in $\mathbb{R}^m$ or distribution of compact support in which case the integral is understood in the duality sense.

The FBI transform is a powerful tool in the analysis of partial differential equations and microlocal analysis, particularly for studying solutions with specific types of singularities or asymptotic behaviors. It generalizes the classical Fourier transform by allowing for more flexible phase functions and can handle distributions that are not well-behaved under the classical Fourier framework (see Adwan and Hoepfner (2010), Adwan and Hoepfner (2015), Berhanu and Hailu (2017) and Berhanu and Hounie (2012)). Both the classical Fourier transform and the FBI transform serve as foundation tools in modern applied mathematics, each with unique strengths suited to different types of analyses.

In Berhanu and Hounie (2012) the authors introduced the following more general class of FBI transforms. Let $\psi \in \mathcal{S}(\mathbb{R}^m)$ such that $0 \neq \int |\psi(x)|dx < \infty$. They define the FBI transform of a compactly supported distribution $u \in \mathcal{E}'(\mathbb{R}^m)$ with generating function ψ and parameter λ $(0 < \lambda < 1)$, by

$$\mathcal{F}_{\psi,\lambda}u(x,\xi) = \langle u(x'), e^{i\xi.(x-x')}\psi\Big(|\xi|^\lambda(x-x')\Big)\rangle, \tag{1.2}$$

where the right hand side is understood in the duality sense. A simple example of this generalized transform (1.2) includes the class

$$\mathcal{F}_k u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-x')-|\xi||x-x'|^{2k}} u(x')dx',\ x,\xi \in \mathbb{R}^m,\ k = 1,2,\ldots \tag{1.3}$$

When $k = 1$, the transform (1.3) reduces to the classical FBI transform (1.1). In Berhanu and Hounie (2012)(Theorem 5.1) the authors gave an application where the classic FBI transform cannot be used but one with a power of four can be used to characterize the analytic wave front set. Here we will present an application where the FBI transform with phase power of eight can be used to characterize the $s-$ wave front set of a distribution but the FBI transform with phase powers 2, and 4 can not be used.

Objective of the study:
The objective of the study is to provide an application of the more generalized FBI transform in solving problems related to web front sets.

The structure of this paper is as follows: Section 1 deals with introduction. Section 2 presents the definition of Gevery functions and Gevrey Wave front sets. In Section 3, we construct an example of a problem in which the FBI transform with phase power of eight can be used to characterize the $s-$ wave front set of a distribution but the FBI Transform with phase powers 2 and 4 can not be used.

2 Preliminaries

This section provides essential definitions and concepts that will be used later.

Definition 2.1. Let $s \geq 1$, and $f(x) \in C^\infty(\Omega)$, where $\Omega \subset \mathbb{R}^m$ open. Then a function f is a Gevrey function of order s on Ω if for any compact subset K of Ω there is a constant $C_K > 0$ such that

$$|\partial^\alpha f(x)| \leq C_K^{|\alpha|+1}(\alpha!)^s, \forall \alpha \in \mathbb{N}_0^n \, , \forall x \in K.$$

We denote the class of Gevrey functions of order s on Ω by $G^s(\Omega)$.

Example 2.2. For $s > 1$ define $f_s(t)$ by

$$f_s(t) = \begin{cases} e^{-t^{-\frac{1}{s-1}}}, \ t > 0 \\ 0, \ t \leq 0 \end{cases}$$

Then $f_s \in G^s(\mathbb{R})$.But $f \notin \ C^\omega(\mathbb{R})$.

Definition 2.3. Let Ω be open, $x_0 \in \Omega$, $s > 1$. We say $(x_0, \xi^0) \notin WF_s(u)$(Gevrey wave front set) if there exist $\phi \in G_0^s(\Omega), \phi = 1$ near x_0, a conic neighborhood Γ of ξ^0 and constants $c_1, c_2 > 0$ such that

$$|\widehat{\phi u}(\xi)| \leq c_1 \exp\big(-c_2|\xi|^{\frac{1}{s}}\big), \ \forall \xi \in \Gamma.$$

The following theorem gives the characterization of Gevrey $(s-)$ wave front sets using the FBI transform.

Theorem 2.4 (Berhanu and Hailu (2017), Theorem 3.2)**.** *Let $u \in \mathcal{E}'(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m$ with $|\xi^0| = 1$. Then $(x_0, \xi^0) \notin WF_s(u), s > 1$ if and only if there exist a neighborhood V of x_0, a conic neighborhood Γ of ξ^0 and constants $a, b > 0$ such that for some $\phi \in C_0^\infty(\mathbb{R}^m), \phi \equiv 1$ near x_0,*

$$|\mathcal{F}(\phi u)(x, \xi)| \leq ae^{-b|\xi|^{\frac{1}{s}}}, \ \ (x, \xi) \in V \times \Gamma.$$

3 Application of the generalized fbi transform

Let's first establish some notations and inequalities for later use.

For $p \in \mathbb{N}$, $z \in \mathbb{C}^m$, $z_j = x_j + iy_j$, we will use the notation $z^{2p} = \left(\sum_{j=1}^m z_j^2\right)^p$. Then

$$\begin{aligned} z^{2p} &= \left(\sum_{j=1}^m z_j^2\right)^p \\ &= (|x|^2 - |y|^2 + 2i(x.y))^p \\ &= \sum_{j=0}^p \binom{p}{j} (|x|^2 - |y|^2)^{p-j} (2i)^j (x.y)^j \\ &= \sum_{j=0}^p \binom{p}{2j} (-1)^j (|x|^2 - |y|^2)^{p-2j} (2(x.y))^{2j} \\ &+ \sum_{j=0}^p \binom{p}{2j+1} (-1)^j (|x|^2 - |y|^2)^{p-(2j+1)} (2(x.y))^{2j+1}. \end{aligned}$$

where we have used the convention that $\binom{p}{2j} = 0$ if $2j > p$ and similarly for $\binom{p}{2j+1} = 0$ if $2j+1 > p$.
Thus

$$\Re z^{2p} = \sum_{j=0}^p \binom{p}{2j} (-1)^j (|x|^2 - |y|^2)^{p-2j} (2(x.y))^{2j}.$$

Here we will consider only the cases $p = 1, 2, 3, 4$. Thus in particular,

$$\Re z^2 = |x|^2 - |y|^2$$

$$\Re z^4 = (|x|^2 - |y|^2)^2 - 4(x.y)^2 \tag{3.1}$$

$$\Re z^6 = (|x|^2 - |y|^2)^3 - 12(|x|^2 - |y|^2)(x.y)^2$$

$$\Re z^8 = (|x|^2 - |y|^2)^4 - 24(x.y)^2(|x|^2 - |y|^2)^2 + 16(x.y)^4 \tag{3.2}$$

Lemma 3.1. *For $p \in \mathbb{N}$, $z \in \mathbb{C}^m$, $z_j = x_j + iy_j$, the following inequalities holds true.*

$$\Re z^8 \geq |x|^8 - 28|x|^6|y|^2 - 26|x|^2|y|^6 \tag{3.3}$$

and

$$\Re z^8 \geq -144|x|^4|y|^4 \tag{3.4}$$

Proof. From (3.2), we have

$$\begin{aligned}
\Re z^8 &= (|x|^2-|y|^2)^4 - 24(x.y)^2(|x|^2-|y|^2)^2 + 16(x.y)^4 \\
&\geq (|x|^2-|y|^2)^4 - 24|x|^2|y|^2(|x|^2-|y|^2)^2 \\
&= (|x|^2-|y|^2)^2\Big(|x|^2-|y|^2)^2 - 24|x|^2|y|^2\Big) \\
&= (|x|^2-|y|^2)^2\Big(|x|^4-2|x|^2|y|^2+|y|^4 - 24|x|^2|y|^2\Big) \\
&= (|x|^2-|y|^2)^2\Big(|x|^4-26|x|^2|y|^2+|y|^4\Big) \\
&\geq (|x|^2-|y|^2)^2\Big(|x|^4-26|x|^2|y|^2\Big) \\
&\geq |x|^8 - 28|x|^6|y|^2 - 26|x|^2|y|^6
\end{aligned}$$

Also we have

$$\begin{aligned}
\Re z^8 &= (|x|^2-|y|^2)^4 - 24(x.y)^2(|x|^2-|y|^2)^2 + 16(x.y)^4 \\
&\geq (|x|^2-|y|^2)^4 - 24(x.y)^2(|x|^2-|y|^2)^2 \\
&\geq (|x|^2-|y|^2)^4 - 24|x|^2|y|^2(|x|^2-|y|^2)^2 \\
&= \Big((|x|^2-|y|^2)^2 - 12|x|^2|y|^2\Big)^2 - 144|x|^4|y|^4 \\
&\geq -144|x|^4|y|^4.
\end{aligned}$$

□

Let $U \subset \mathbb{R}^m$ be a bounded neighborhood of $0 \in \mathbb{R}^m$, $0 \in I \subset \mathbb{R}$ be an open interval and $t^* \in I^+ = I \cap (0, +\infty)$. Consider $\phi := (\phi_1, \ldots, \phi_m) \in (G^s(I))^m$ and the vector field L defined by

$$L \doteq \frac{\partial}{\partial t} - i\sum_{j=1}^{m} \frac{\partial \phi_k}{\partial t}(t)\frac{\partial}{\partial y_k}, \quad k = 1, \ldots, m.$$

If $Z_k(y,t) = y_k + i\phi_k(t)$, then $LZ_k = 0$ and $Z_k(y,0) = y_k$ $(k = 1, \ldots, m)$. Denote $Z \doteq (Z_1, \ldots, Z_m)$.

Let $B_r(0) = \{x \in \mathbb{R}^m : |x| < r\}$ for some $r > 0$. Set $\Omega = B_r(0) \times I$. If $h(x,t)$ is a solution in Ω of the equation

$$Lh = 0,$$

we are interested in the s-wave front set of the solution $h_0(x) = h(x,0)$. We will assume that $\phi(0) = 0$.

Theorem 3.2. *Let U, I, I^+ and ϕ be as above. Let $\xi^0 \in \mathbb{R}^m \setminus \{0\}$ with $|\xi^0| = 1$. For some $\epsilon > 0$, suppose that*

$$\begin{aligned}
&1) \ -\phi(t^*).\xi^0 \geq \epsilon^{13} \\
&2) \ |\phi(t)| \leq \epsilon^4, \ \forall t \in [0, t^*]
\end{aligned}$$

Then there exists $\epsilon_0 > 0$ *such that if* $0 < \epsilon \leq \epsilon_0$ *and* h *is a solution of (3) in* $\Omega = B_r(0) \times I$, *then* $(0, \xi) \notin WF_s(h_0)$.

Proof. We will follow the the techniques used in Berhanu and Hounie (2012). Shrinking $B_r(0)$ and taking $\epsilon > 0$ small, with out loss of generality assume $r^2 = \epsilon$. Let $g \in G_0^s(B_r(0)), g(y) \equiv 1$ for $|y| \leq \frac{r}{2}$. Let

$$F(x,t,\xi) = \int_{\mathbb{R}^m} e^{Q(x,y,t,\xi)} g(y) h(y,t) dy,$$

where

$$Q(x,y,t,\xi) = i\xi.(x - Z(y,t)) - K|\xi|(x - Z(y,t))^8,$$

where $K > 0$ will be determined latter. For $z \in \mathbb{C}^m$ we have used the notation

$$z^8 = \left(\sum_{j=1}^m z_j^2 \right)^4.$$

Let

$$\begin{aligned} I(x,\xi) &= \int_{\mathbb{R}^m} \int_0^{t^*} e^{Q(x,y,t,\xi)} L(g(y)h(y,t)) dt dy \\ &= \int_{\mathbb{R}^m} \int_0^{t^*} e^{Q(x,y,t,\xi)} [h(y,t)L(g(y)) + g(y)L(h(y,t))] dt dy \\ &= \int_{\mathbb{R}^m} \int_0^{t^*} e^{Q(x,y,t,\xi)} h(y,t) L(g(y)) dt dy. \end{aligned} \tag{3.5}$$

Then integration by parts and the fact that $LZ_k = 0$ gives

$$I(x,\xi) = F(x,t^*,\xi) - F(x,0,\xi). \tag{3.6}$$

Now choosing $p(x) = |x|^8$ and $\Psi(x) = e^{-|\xi|p(x)}$, we have

$$\mathcal{F}(gh_0)(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)} e^{-|\xi|p(x-y)} g(y) h(y,0) dy = F(x,0,\xi).$$

Thus to prove the theorem, it suffice to show that for some constants $C_1, C_2 > 0$,

$$|F(x,0,\xi)| \leq C_1 e^{-C_2|\xi|^{1/s}}, \ (x,\xi) \text{ in a conic neighborhood of } (0,\xi^0).$$

Let $E(x,y,t,\xi) = -\Re Q(x,y,t,\xi)$. For $x = 0$ and $\xi = \xi^0$, $|\xi^0| = 1$, we have

$$\begin{aligned} E(0,y,t,\xi^0) &= -\Re Q(0,x,y,t,\xi^0) \\ &= -\Re\Big(i\xi^0.(-Z(y,t)) - K|\xi^0|(-Z(y,t))^8 \Big) \\ &= -\xi^0.\phi(t) + K\Re\Big(Z(y,t) \Big)^8. \end{aligned}$$

Therefore, using (3.4)

$$\Re\Big(Z(y,t) \Big)^8 \geq -144|y|^4|\phi(t)|^4$$

and thus

$$E(0,y,t,\xi^0) \geq -\xi^0.\phi(t) + K(-144|y|^4|\phi(t)|^4). \tag{3.7}$$

Now since $h(y,t)$ is a solution, and $g \equiv 1$ for $|y| \leq \frac{r}{2}$, in the integrand of $F(x,t,\xi)$, the term $L(g(y)h(y,t))$ is supported in the ball $|y| \leq \frac{r}{2}$. We will estimate the quantity $E(0,y,t,\xi^0)$ when $\frac{r}{2} < |y| \leq r$.
Thus in particular when $t = t^*$ and $\frac{r}{2} < |y| \leq r$, $r^2 = \epsilon$ by assumptions of the theorem and (3.7we have

$$\begin{aligned} E(0,y,t^*,\xi^0) &\geq \epsilon^{13} + K(-144\epsilon^{16}|y|^4) \\ &\geq \epsilon^{13} - \frac{144}{4}K\epsilon^{18} \\ &= \epsilon^{13}(1 - 36K\epsilon^5). \end{aligned} \tag{3.8}$$

For $r^2 = \epsilon$, $\frac{r}{2} \leq |y| \leq r$ using (3.3) and the assumptions, for any $\epsilon > 0$ sufficiently small such that $28\epsilon^7 + 26\epsilon^{21} < \frac{1}{2^9}$, we have:

$$\begin{aligned} E(0,y,t,\xi^0) &\geq -\xi^0.\phi(t) + K\left(|y|^8 - 28\epsilon^8|y|^6 - 26|y|^2|\phi(t)|^6\right) \\ &\geq -\epsilon^4 + K\left(|y|^8 - 28\epsilon^8|y|^6 - 26|y|^2\epsilon^{24}\right) \\ &\geq -\epsilon^4 + K\left(\frac{r^8}{2^8} - 28r^6\epsilon^8 - 26r^2\epsilon^{24}\right) \\ &= -\epsilon^4 + K\left(\frac{\epsilon^4}{2^8} - 28\epsilon^{11} - 26\epsilon^{25}\right) \\ &= -\epsilon^4 + \epsilon^4 K\left(\frac{1}{2^8} - 28\epsilon^7 - 26\epsilon^{21}\right) \\ &\geq \epsilon^4\left(\frac{K}{2^9} - 1\right). \end{aligned}$$

Thus for $K \geq 2^{10}$, we have

$$E(0,y,t,\xi^0) \geq \epsilon^4. \tag{3.9}$$

Thus if we choose $K = 2^{10}$ to grant (3.9) and pick $\epsilon_0 < \left(\frac{1}{36(2^{10})}\right)^{\frac{1}{5}}$, for any $0 < \epsilon < \epsilon_0$ the right hand side of (3.8) will be positive. Using the fact that $E(x,t,y,\xi)$ is homogeneity of of degree one in ξ, (3.8) and (3.9) imply that there is a neighborhood V of the origin in $\mathbb{R}^m$, and an open cone $\Gamma \subset \mathbb{R}^m$ centered at ξ^0 such that for some $C > 0$

$$E(x,y,0,\xi) \geq C|\xi|,\ \forall (x,\xi) \in V \times \Gamma, y \in \text{ supp } g \tag{3.10}$$

and

$$E(x,y,t,\xi) \geq C|\xi|,\ \forall (x,\xi) \in V \times \Gamma, 0 \leq t \leq t^*, \frac{\epsilon^{\frac{1}{2}}}{2} \leq |y| \leq \epsilon^{\frac{1}{2}}. \tag{3.11}$$

Therefore, from (3.5),(3.6), (3.10) and (3.11), we have

$$\begin{aligned}
|\mathcal{F}(gh_0)(x,\xi)| &= |F(x,0,\xi)| \\
&= |F(x,t^*,\xi) - I(x,\xi)| \\
&\leq |F(x,t^*,\xi)| + |I(x,\xi)| \\
&= \left| \int_{\mathbb{R}^m} \int_0^{t^*} e^{Q(x,y,t,\xi)} h(y,t) Lg(y) dt dy \right| + \left| \int_{\mathbb{R}^m} e^{Q(x,y,t^*,\xi)} g(y) h(y,t) dt dy \right| \\
&\leq \int_{\mathbb{R}^m} \int_0^{t^*} e^{-C|\xi|} \Big| h(y,t) Lg(y) \Big| dy dt + \int_{\mathbb{R}^m} e^{-C|\xi|} \Big| g(y) h(y,t) \Big| dt dy \\
&\leq Be^{-C|\xi|} \leq Be^{-C|\xi|^{\frac{1}{s}}}, \quad (x,\xi) \in V \times \Gamma.
\end{aligned}$$

□

Remark 3.3. We now show here that it is not possible to prove the preceding theorem with the classical FBI transform (quadratic phase), and the FBI transform with phase powers of four.
a) Let

$$F_1 = \int_{\mathbb{R}^m} e^{Q_1(x,y,t,\xi)} g(y) h(y,t) dy,$$

where

$$Q_1(x,y,t,\xi) = i\xi.(x - Z(y,t)) - K|\xi|(x - Z(y,t))^2$$

for some $K > 0$ fixed. Here for $z \in \mathbb{C}^m$, we are using the notation $z^2 = \sum_{j=1}^m z_j^2$. Assuming $|\xi^0| = 1$ and letting $E_1(x,y,t,\xi) = -\Re Q_1(x,y,t,\xi)$ we have

$$E_1(x,y,t^*,\xi^0) = -\xi^0.\phi(t^*) + K|\xi^0|(|x-y|^2 - |\phi(t^*|^2).$$

Suppose that $-\xi^0.\phi(t^*) = \epsilon^{13}$ and $|\phi(t^*| = \epsilon^4$.Then

$$E_1(x,y,t^*,\xi^0) = \epsilon^{13} + K(|x-y|^2 - \epsilon^8).$$

In particular

$$E_1(x,x,t^*,\xi^0) = \epsilon^{13} - K\epsilon^8 = \epsilon^8(\epsilon^5 - K)$$

which is negative if $K > \epsilon^5$.
Suppose $K < \epsilon^5$. Since $E_1(x,y,t,\xi^0) = -\phi(t) \cdot \xi^0 + K[|x-y|^2 - |\phi(t)|^2]$, we have

$$\begin{aligned}
E_1(0,y,t,\xi^0) &= -\phi(t) \cdot \xi^0 + K[|y|^2 - |\phi(t)|^2] \\
&\leq -\phi(t) \cdot \xi^0 + K|y|^2
\end{aligned}$$

Thus for a point t such that $-\phi(t) \cdot \xi^0 = -\epsilon^4$ and for $|y| \geq \frac{r}{2}$ we obtain

$$\begin{aligned}
E_1(0,y,t,\xi^0) &\leq -\phi(t) \cdot \xi^0 + K|y|^2 \\
&< -\epsilon^4 + \epsilon^5 r^2.
\end{aligned}$$

But then since ϵ is small, $-\epsilon^4 + \epsilon^5 r^2$ is negative unless r is large. Thus the quantity $E_1(x,y,t,\xi^0)$ may be negative in this case also.
Therefore, the FBI transform with quadratic phase can not be used to solve the

problem.
b) Let

$$F_2 = \int_{\mathbb{R}^m} e^{Q_2(x,y,t,\xi)} g(y) h(y,t) dy,$$

where

$$Q_2(x,y,t,\xi) = i\xi.(x - Z(y,t)) - K|\xi|(x - Z(y,t))^4$$

for some $K > 0$ fixed. Here for $z \in \mathbb{C}^m$, we are using the notation

$$z^4 = \left(\sum_{j=1}^{m} z_j^2 \right)^2.$$

From (3.1), we have that

$$\Re z^4 = (|x|^2 - |y|^2)^2 - 4(x.y)^2$$

Assuming as before that $E_2(x,y,t,\xi) = -\Re Q_2(x,y,t,\xi)$ we have

$$\begin{aligned} E_2(x,y,t^*,\xi^0) &= -\Re\Big(i\xi \cdot (x - y - i\phi) + K|\xi|(x - y - i\phi)^4 \Big|) \\ &= -\xi \cdot \phi + K|\xi| \Re(x - y - i\phi)^4) \\ &= -\xi \cdot \phi + K|\xi| \Big((|x-y|^2 - |\phi|^2|) - 4((x-y)\cdot\phi)^2 \Big). \end{aligned}$$

Thus, for $|\xi^0| = 1$,

$$\begin{aligned} E_2(0,y,t,\xi^0) &= -\xi^0 \cdot \phi + K(|y|^4 - 2|y|^2|\phi|^2 + |\phi|^4 - 4(\phi \cdot y)^2 \\ &\leq -\xi^0 \cdot \phi + K(|y|^4 - 6|y|^2|\phi|^2| + |\phi|^4) \text{ using Schewarz inequality.} \end{aligned}$$

Thus for $-\xi^0.\phi(t) = \epsilon^{13}$, $|\phi(t)| = \epsilon^4$, $t = t^*$ we have

$$\begin{aligned} E_2(0,y,t^*,\xi^0) &\leq -\epsilon^{13} + K(|y|^4 - 2|y|^2|\epsilon^8 + \epsilon^{16} \\ &\leq \epsilon^{13} - 8K\epsilon^{16}. \end{aligned}$$

since $s^4 - 6s^2\epsilon^8 + \epsilon^{16}$ attains minimum at $-8\epsilon^{16}$.
Hence, for $K > \frac{1}{8\epsilon^3}$, $E_2(x,y,t,\xi^0)$ becomes negative.
Moreover, if $K < (2\epsilon)^{-3}$, when $|y| \geq r/2$, the term $E_2(x,y,t,\xi^0)$ again becomes negative, for instance when $\sup_t \phi(t) \cdot \xi^0 \geq 0$. Therefore, the FBI transform with phase power of 4 can not be used to solve the problem.

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