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Research article

Existence of solutions for nonlinear elliptic PDEs with variable exponents

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Abstract

In this paper, we prove the existence of solutions for a particular class of Dirichlet boundary value problems of the following type using the variational framework offered by the Mountain Pass Theorem:

$$\begin{cases} b(u) - \triangle_{p(x)}(u) = f(x,u), \text{in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases},$$

where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $p(x)$ is a positive continuous function and $\triangle_{p(x)}u(x) = \text{div}(|\nabla u(x)|^{p(x)-2}\nabla u(x))$. The function b is continuous and non decreasing real-valued function, $f(x,u)$ satisfies some given conditions and $p : \overline{\Omega} \to \mathbb{R}^+$ is continuous function such that $p(x) > 1$.

Keywords: Energy functional, Lebesgue-Sobolev spaces, mountain pass theorem, variable exponent, weak solution.

1 INTRODUCTION

The study of elliptic equations with variable-exponent growth conditions has become a significant area of research during the last ten years. The limitations of classical Lebesgue and Sobolev spaces $L^p, W^{1,p}$ for modelling media with spatial inhomogeneities are reflected in this trend: The behaviour of electrorheological fluids, some models in nonlinear elasticity, and high-fidelity image-restoration tasks are frequently not captured by constant-exponent frameworks. The generalised Lebesgue spaces $L^{p(x)}$ and Sobolev spaces $W^{m,p(x)}$, whose functional-analytic foundations have been thoroughly developed, naturally result from treating the integrability exponent as a function $p(x)$. For the existence, regularity, and qualitative analysis of solutions to the resulting nonstandard elliptic problems, these variable-exponent spaces offer the proper variational setting. The works Nakano (1950), Fan et al. (2005), Tsenov (1961), Musielak (1983), Fan and Zhao (2001), and Ruzicka (2000) contain important characteristics of these spaces. Rădulescu (2015) and Harjulehto et al. (2010) have conducted a significant survey on Lebesque and Sobolev spaces with variable exponent. A crucial

mathematical framework for explaining complicated physical phenomena, especially those that take place in materials with notable internal inhomogeneities, is nonlinear elliptic partial differential equations with variable exponents.
In this piece, we focus on the equation of the type:

$$\begin{cases} b(u) - \Delta_{p(x)} u = f(x,u) & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases}. \tag{1.1}$$

The $p(x)$-Laplacian, which is the divergence of $(|\nabla u(x)|^{p(x)-2}\nabla u(x))$, is combined with the operator $b(u)$, which represents a continuous and non-decreasing real-valued function. A bounded domain $\Omega \subset \mathbb{R}^n$ is used to analyse this structure, provided that the solution u vanishes on the boundary $\partial\Omega$. In order to maintain non-standard but mathematically tractable growth conditions, it is assumed that the exponent $p(x)$ is a continuous function on the closure of the domain $\overline{\Omega}$, with $p(x) > 1$ throughout the domain.
The problem (1.1) can be regarded as an extension of the problem analysed by Konaté and Ouaro (2016)

$$\begin{cases} b(u) - \sum_{i=1}^{N} \dfrac{\partial}{\partial x_i} a_i\left(x, \dfrac{\partial u}{\partial x_i}\right) = \mu & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega, \end{cases} \tag{1.2}$$

where b is a continuous, surjective and non-decreasing function and μ is a bounded Radon diffuse measure. They proved the existence and uniqueness of entropy solution to the problem by approximation method.

EL. Amrouss and Kissi (2013), studied the same problem and using adequate variational formulation and a variant of Mountain Pass theorem, they proved the existence of multiple solutions. The weak solutions of the class of elliptic equations under the nonstandard growth condition were also studied in Adamowicz and Toivanen (2015); Fan and Zhao (2000). The purpose of this paper is to extend the problem studied in EL. Amrouss and Kissi (2013) by an input defined by a nondecreasing positive function $b(u)$ and investigate the existence of its solutions.
Objective of the study:
The objective of the study is to provide a proof of the existence of solutions of a class of Dirichlet boundary value problems of nonlinear PDEs with variable exponents.
The paper is organized as follows: Section 2 reviews the fundamental concepts of variable exponent spaces, while Section 3 outlines the necessary assumptions and preliminary results. The main theorem is introduced in Section 4, followed by its formal proof in Section 5 and conclusion in Section 6.

2 PRELIMINARIES: VARIABLE EXPONENT SPACES

Let $\Omega \subset \mathbb{R}^n$ be a Lebesgue measurable such that $|\Omega| > 0$. Let $P(\Omega)$ denote the family of all measurable functions $p : \overline{\Omega} \to [1, \infty]$. For $p \in P(\Omega), \Omega_\infty^p = \Omega_\infty = \{x \in \Omega : p(x) = \infty\}$. Denote $p^- = ess\inf_{x\in\Omega} p(x)$ and $p^+ = ess\sup_{x\in\Omega} p(x)$. The associated variable exponent Lebesgue space $L^{p(x)}$ is defined by (see Fan and Zhao (2001))

$$L^{p(x)} = \{u; u \in \mathbb{R} : \int_\Omega |u(x)|^{p(x)} dx < \infty\}. \tag{2.1}$$

The associated Luxembourg norm is then given by

$$|u|_{p(x)} = inf(\lambda > 0 : \int_\Omega |\frac{u(x)}{\lambda}|^{p(x)} dx \leq 1).$$

Note that variable exponent Lebesgue spaces are separable and reflexive Banach Spaces. If $0 < |\Omega| < \infty$ and p_1, p_2 are variable exponents such that $p_1(x) \leq p_2(x)$ almost everywhere in Ω, then there exists a continuous embedding $L^{p_2(x)} \to L^{p_1(x)}$. We denote by $L^{p'(x)}(\Omega)$ the conjugate space of $L^{p(x)}(\Omega)$, where $\frac{1}{p(x)} + \frac{1}{p'(x)} = 1$. For $u \in L^{p(x)}(\Omega)$ and $v \in L^{p'(x)}(\Omega)$, the Hölder inequality holds true:

$$|\int_\Omega uvdx| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)||u||_{p(x)}||v||_{p'(x)} \leq 2||u||_{p(x)}||v||_{p'(x)}. \tag{2.2}$$

The transition from constant to variable exponents depends on the definition of the convex functional called the modular, which takes the place of the conventional p-th power integral. $\rho p(x) : L^{p(x)}(\Omega) \to \mathbb{R}$ defined by

$$\rho p(x)(u) = \int_\Omega |u(x)|^{p(x)} dx.$$

When $p^+ = +\infty$, the modular is defined by

$$\rho p(x)(u) = \int_{\Omega\setminus\Omega_\infty} |u(x)|^{p(x)} dx + \sup_{x\in\Omega_\infty} |p(x)|.$$

Convergence with respect to the modular is equivalent to that with respect to the norm. Moreover, the following relation between the modular and norm holds:

$$\min\{||u||_{p(\cdot)}^{p^-}, ||u||_{p(\cdot)}^{p^+}\} \leq \int_\Omega |u(x)|^{p(x)}\, dx \leq \max\{||u||_{p(\cdot)}^{p^-}, ||u||_{p(\cdot)}^{p^+}\}.$$

We define the variable exponent Sobolev Space by

$$W^{1,p(\cdot)}(\Omega) = \{u \in L^{p(\cdot)}(\Omega) : |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

On $W^{1,p(\cdot)}$, we may consider one of the following equivalent norms

$$||u||_{p(\cdot)} = ||u||_{L^{p(\cdot)}(\Omega)} + ||\nabla u||_{L^{p(\cdot)}(\Omega)},$$

or

$$||u|| = \inf\Big\{\lambda > 0 : \int_\Omega \Big(\big|\tfrac{u(x)}{\lambda}\big|^{p(x)} + \big|\tfrac{\nabla u(x)}{\lambda}\big|^{p(x)}\Big)\, dx \leq 1\Big\}.$$

We also define $W_0^{1,p(\cdot)}(\Omega)$ as the closure of $C_0^\infty(\Omega)$ in $W^{1,p(\cdot)}(\Omega)$.
The generalized Lebesgue–Sobolev space $W^{m,p(\cdot)}(\Omega)$ is defined by

$$W^{m,p(\cdot)}(\Omega) = \{\, u \in L^{p(\cdot)}(\Omega) : D^\alpha u \in L^{p(\cdot)}(\Omega),\ |\alpha| \le m \,\},$$

where m is a positive integer and $p(\cdot) \in L_+^\infty(\Omega)$. These spaces are equipped with the norm $||u||_{W^{m,p(\cdot)}(\Omega)}$ defined by

$$||u||_{W^{m,p(\cdot)}(\Omega)} = \sum_{|\alpha|\le m} ||D^\alpha u||_{L^{p(\cdot)}(\Omega)}.$$

Important function space properties are satisfied by the function spaces $W^{1,p(\cdot)}(\Omega)$ and $W^{m,p(x)}(\Omega)$. The reader is recommended to refer Fan and Zhao (2001) and EL. Amrouss and Kissi (2013) for these properties.

3 ASSUMPTIONS AND PRELIMINARY RESULTS

Consider the problem (1.1) and the associated eigen value problem:

$$\begin{cases} b(u) - \triangle_{p(x)}(u) = \boldsymbol{\lambda}|u|^{r(x)-2}u, \text{ in } \Omega \\ u = 0, \text{ on } \partial\Omega \end{cases}. \tag{3.1}$$

We assume the following conditions on the functions b and f:

(A1): Ω is a bounded connected open set of $\mathbb{R}^n$, $n \ge 1$.

(A2): The function $b : \mathbb{R} \to \mathbb{R}$ is continuous and non-decreasing real-valued function.

(A3): The functions $p, r : \overline{\Omega} \to \mathbb{R}^+$ are continuous such that $p(x) > 1$ and $r(x) > 1$. Then, we have

$$\begin{cases} p^- = \operatorname{ess\,inf}_\Omega p(x) > 1, & r^- > 1, \\ p^+ = \operatorname{ess\,sup}_\Omega p(x) < \infty, & r^+ < \infty. \end{cases} \tag{3.2}$$

(A4): The reaction term $f(x,u)$ is a Carathéodory function with the growth condition

$$|f(x,\xi)| \le C\big(j(x) + |\xi|^{r(x)-1}\big),$$

where $j(x) \in L^{r'(\cdot)}(\Omega)$, $C > 0$ with

$$\frac{1}{r(x)} + \frac{1}{r'(x)} = 1.$$

(A5): $u(x) \in L^{p(x)}$, $b(u) \in L^{p'(x)}$, $\nabla u(x) \in \big(L^{p(x)}\big)^n$.

Under Assumptions (A1)–(A5), the boundary-value problem (3.1) has at least one weak solution $u \ne 0$, $u \in W_0^{1,p(x)}(\Omega)$, defined by

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u(x)\cdot\nabla v(x)\,dx + \int_\Omega \big[b(u) - f(x,u)\big]v\,dx = 0, \quad \forall v \in W_0^{1,p(x)}(\Omega),$$

where v is the test function. Recall that the weak solution of the problem (1.1) is a critical point of the associated energy functional defined by

$$J(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_\Omega G(x,u)\, dx. \tag{3.3}$$

where

$$G(x,u) = \int_0^u \big[b(\xi) - f(x,\xi)\big]\, d\xi.$$

For $b(0) = 0$ and $f(x,0) = 0$, then $u = 0$ is the solution of the BVP (1.1). However, we are interested in nontrivial solution. We know that solutions of (1.1) are critical points of the functional (3.3).
Now, write J as

$$J(u) = J_1(u) + J_2(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx + \int_\Omega G(x,u)\, dx.$$

Then

$$J_1'(u) = \frac{d}{du} \int_\Omega \frac{1}{p(x)} \left(\sum_{j=1}^n |\partial_j u|^2 \right)^{\frac{p(x)}{2}} dx = \int_\Omega \frac{1}{p(x)} \frac{d}{du} \left(\sum_{j=1}^n |\partial_j u|^2 \right)^{\frac{p(x)}{2}} dx$$

$$= \int_\Omega \frac{1}{p(x)} \frac{p(x)}{2} \left(\sum_{j=1}^n |\partial_j u|^2 \right)^{\frac{p(x)}{2}-1} 2 \left(\sum_{j=1}^n |\partial_j u| \right) dx$$

$$= \int_\Omega \left(\sum_{j=1}^n |\partial_j u|^2 \right)^{\frac{p(x)-2}{2}} \left(\sum_{j=1}^n |\partial_j u| \right) dx = \int_\Omega |\nabla u|^{p(x)-2} \nabla u(x)\, dx.$$

Thus, the energy functional $J(u)$ will act on the generalized Sobolev space $W_0^{1,p(x)}(\Omega)$. It is known that under assumption (A4), J is well defined and its C^1 derivative given by

$$\langle J_1'(u), v\rangle = \int_\Omega |\nabla u|^{p(x)-2} \nabla u(x) \cdot \nabla v(x)\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

$$\langle J_2'(u), v\rangle = \int_\Omega \big[b(u) - f(x,u)\big] v\, dx, \quad \forall v \in W_0^{1,p(x)}(\Omega).$$

To prove the Gâteaux differentiability of the functional $J_1(u)$, denote

$$|\nabla u|^{p(x)-2}\nabla u = \alpha(x, \nabla u), \qquad \frac{1}{p(x)} |\nabla u|^{p(x)} = A(x, \nabla u),$$

so that

$$\alpha(x,\xi) = \nabla_\xi A(x,\xi).$$

Again we assume the following conditions on the differential operators α and A:

(B1): $|\alpha(x,\xi)| \le c_1\big(1 + |\xi|^{p(x)-1}\big)$.
(B2): $0 \le \big(\alpha(x,\xi) - \alpha(x,\psi)\big) \cdot (\xi - \psi)$.
(B3): $A\big(x, \frac{\xi+\psi}{2}\big) \le \frac{1}{2}A(x,\xi) + \frac{1}{2}A(x,\psi) - k|\xi - \psi|^{p(x)}, \quad k > 0$.
(B4): $|\xi|^{p(x)} \le \alpha(x,\xi) \cdot \xi \le p(x) A(x,\xi)$.

(B5): $A(x,0)=0$ and $|A(x,\xi)| \leq |\xi| + |\xi|^{q(x)-1}$.

Let $u, v \in W_0^{1,p(x)}(\Omega)$. Fix $x \in \Omega$ and $0 < |r| < 1$. By the mean value theorem, there exists $\nu \in [0,1]$ such that

$$\frac{|A(x,\nabla u + r\nabla v) - A(x,\nabla u)|}{|r|} = |\alpha(x,\nabla u + \nu r \nabla v)|\,|\nabla v|. \tag{3.4}$$

Now. using (B1) and (B2), we obtain

$$\begin{aligned}\frac{\big|A(x,\nabla u(x) + r\nabla v(x)) - A(x,\nabla u(x))\big|}{r} &\leq \big[c_1 + c_1\big(|\nabla u(x)| + |\nabla v(x)|^{p(x)-1}\big)\big]|\nabla v(x)| \\ &\leq \big[c_1 + c_1 2^{p^+}\big(|\nabla u(x)|^{p(x)-1} + |\nabla v(x)|^{p(x)-1}\big)\big]|\nabla v(x)| \\ &\leq c_1\Big[1 + 2^{p^+}\big(|\nabla u(x)|^{p(x)-1} + |\nabla v(x)|^{p(x)-1}\big)\Big]|\nabla v(x)|.\end{aligned}$$

By Hölder inequality, we get

$$\int_\Omega c_1|\nabla v(x)|\,dx \leq ||c_1||_{L^{p'(x)}}\,||\nabla v(x)||_{L^{p(x)}}. \tag{3.5}$$

and

$$\int_\Omega |\nabla u|^{p(x)-1}|\nabla v(x)|\,dx \leq |||\nabla u|^{p(x)-1}||_{L^{p'(x)}}\,||\nabla v(x)||_{L^{p(x)}}. \tag{3.6}$$

Therefore, from (3.5) and (3.6), we obtain

$$c_1\Big[1 + 2^{p^+}\big(|\nabla u(x)|^{p(x)-1} + |\nabla v(x)|^{p(x)-1}\big)\Big]|\nabla v(x)| \in L^1(\Omega).$$

And from the Lebesgue theorem it follows that

$$\langle J_1'(u), v\rangle = \int_\Omega \alpha(x,\nabla u)\cdot\nabla v\,dx.$$

The established Gâteaux derivative is continuous. In fact, assume that $u_n \to u$ in $W_0^{1,p(x)}(\Omega)$. Using Proposition 2.2 in Fan and Zhao (2000) and assumption (B2), we have

$$\alpha(x,u_n) \to \alpha(x,u) \quad \text{in } \big(L^{q(x)}(\Omega)\big)^n,$$

where

$$q(x) = \frac{p(x)}{p(x)-1}.$$

By the Hölder inequality (2.2), we obtain

$$\big|\langle J_1'(u_n) - J_1'(u), v\rangle\big| \leq ||\alpha(x,u_n) - \alpha(x,u)||_{L^{q(x)}(\Omega)}\,||v||_{L^{p(x)}(\Omega)},$$

and so

$$||J_1'(u_n) - J_1'(u)|| \leq ||\alpha(x,u_n) - \alpha(x,u)||_{L^{q(x)}(\Omega)} \longrightarrow 0 \quad \text{as } n \to \infty.$$

We further assume the following conditions:

(C1):

$$\lim_{|t|\to\infty}\left(G(x,t) - \frac{(p^-)^2}{(p^+)^3}\,\lambda_*|t|^{p^-}\right) = -\infty,$$

uniformly for almost everywhere $x \in \Omega$, where λ_* is the principal eigenvalue of the $p(x)$–Laplacian Dirichlet problem.

(C2): There exist $\mu \in [1, p^-]$ and $\gamma > 0$ such that

$$0 < \mu G(x,t) \le t\,[\,b(t) - f(x,t)\,], \quad \text{for a.e. } x \in \Omega,\ 0 < |t| \le \gamma.$$

(C3): There exist $\theta > p^+$ and $M > 0$ such that

$$|t| \ge M \implies 0 < \theta G(x,t) \le t\,[\,b(t) - f(x,t)\,], \quad \text{for a.e. } x \in \Omega, \text{ and for each } t \in \mathbb{R}.$$

(C4): $b(t) - f(x,t) := g(x,t) = o\Big(|t|^{p^+-1}\Big)$ as $t \to 0$, uniformly for $x \in \Omega$, with $q^- > p^+$.

4 THE MAIN RESULT

We state the main result as follows.

Theorem 4.1 (Existence Theorem)**.** *Assume* (A1),..., (A4), and (C1),..., (C4) are satisfied. Then the problem (1.1) has at least one nontrivial solution.

Before proving the theorem let us discuss and prove some preliminary concepts. Consider a special case of the problem (1.1), where the reaction term has the form $g(x,u) = b(u) - f(x,u) = \lambda |u|^{p(x)-2}u$:

$$\begin{cases} -\Delta_{p(x)} u = \lambda |u|^{p(x)-2}u & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.1}$$

For $u \in W_0^{1,p(x)}(\Omega)$, define two functionals $J_1, J_2 : W_0^{1,p(x)}(\Omega) \to \mathbb{R}$ by

$$J_1(u) = \int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx. \tag{4.2}$$

$$J_2(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx. \tag{4.3}$$

For any $t > 0$, we define

$$M_t = J_2^{-1}(t) = \big\{ u \in W_0^{1,p(x)}(\Omega) : J_2(u) = t \big\}.$$

Then M_t is a submanifold of $W_0^{1,p(x)}(\Omega)$. Since t is a regular value of J_2, put

$$\Sigma_{t,n} = \big\{ H \subset M_t : H = -H,\ \gamma(H) \ge n \big\},$$

where $\gamma(H)$ denotes the genus of H.
Next, we define

$$c_{n,t} = \inf_{H \in \Sigma_{t,n}} \sup_{u \in H} J_1(u), \qquad n = 1, 2, \ldots.$$

The problem (4.1) has infinitely many eigen pairs $\{\lambda(n,t), u(n,t)\}$ such that

$$J_1\big(\pm u(n,t)\big) = t, \qquad J_2\big(\pm u(n,t)\big) = c_{n,t}.$$

with

$$\lambda_{(u,t)} = \frac{\big\langle f\left(u_{(t),t}\right), u_{(t),t} \big\rangle}{\big\langle f_2(u_{(t),t}), u_{(t),t} \big\rangle} \to \infty \quad \text{as} \quad n \to \infty.$$

Next define

$$\bar{\mu}_* = \inf_{u \in W_0^{1,p(x)}(\Omega)} \frac{\int_\Omega |\nabla u|^{p(x)}\, dx}{\int_\Omega |u|^{p(x)}\, dx}, \text{ and } \mu_* = \inf_{u \in W_0^{1,p(x)}(\Omega)} \frac{\int_\Omega \frac{1}{p(x)} |\nabla u|^{p(x)}\, dx}{\int_\Omega \frac{1}{p(x)} |u|^{p(x)}\, dx}, \tag{4.4}$$

and

$$\lambda_* = \inf \Lambda \quad \text{where} \quad \Lambda = \{\lambda \in \mathbb{R} : \lambda \quad \text{is an eigenvalue of the problem (4.1)}\}. \tag{4.5}$$

The following lemma is proved at EL. Amrouss and Kissi (2013).

Lemma 4.2. *Let p^- and p^+ defined by equations (3.2), $\bar{\mu}_*$ and μ_* defined as in equation (4.4), and λ_* defined by equation (4.5). Then*

$$\left(\frac{p^-}{p^+}\right) \bar{\mu}_* \leq \lambda_* \leq \left(\frac{p^+}{p^-}\right) \bar{\mu}_*.$$

Proving the coercivity and lower semicontinuity of the functional $J(u)$ is sufficient to guarantee the existence of a minimizer.

Lemma 4.3. *The functional J is coercive on $W_0^{1,p(x)}(\Omega)$.*

Proof. Let

$$E(x,t) = G(x,t) - \frac{(p^-)^2}{(p^+)^3} A_* |t|^{p^-}.$$

Then, from Assumption (C1), we have that $\forall M > 0$, $\exists R_M > 0$ such that

$$E(x,t) \leq -M, \quad \forall |t| \geq R_M, \quad \text{a.e} \quad x \in \Omega. \tag{4.6}$$

By contradiction, let $k \in \mathbb{R}$ and $u_n \in W_0^{1,p(x)}(\Omega)$ be such that $||u_n|| \to \infty$ and $J(u_n) \leq k$. Putting $v_n = \frac{u_n}{||u_n||}$, one has $v_n \rightharpoonup v_0$ weakly in $W_0^{1,p(x)}(\Omega)$, so $v_n \rightharpoonup v_0$ strongly in $L^{p(x)}(\Omega)$, and hence $v_n(x) \to v_0(x)$ a.e in Ω. Using the definition of J and relation (4.6) it follows that

$$\begin{aligned} k \geq J(u_n) &= \int_\Omega \frac{1}{p(x)} |\nabla u_n|^{p(x)-2}\, dx - \int_\Omega G(x,u_n) dx \\ &\geq \frac{1}{p^+} \int_\Omega |\nabla u_n|^{p(x)-2}\, dx - \frac{(p^-)^2}{(p^+)^3} A_* \int_\Omega |u_n|^p\, dx - \int_\Omega E(x,u) dx \\ &\geq \frac{1}{p^+} \left(||u_n||^{p^-} - \frac{(p^-)^2}{(p^+)^2} A_* \int_\Omega |u_n|^{p^-} dx \right) + M_1, \end{aligned} \tag{4.7}$$

where $M_1 \in \mathbb{R}$. Dividing the last expression of (4.7) by $||u_n||^{p^-}$ and passing to the limit as $n \to \infty$, we get

$$\frac{1}{p^+} \left(1 - \frac{(p^-)^2}{(p^+)^2} A_* \int_\Omega |v_0|^{p-1}\, dx \right) \leq 0. \tag{4.8}$$

Consequently, $v_0 \neq 0$. Let $\Omega_0 = \{x \in \Omega : v_0(x) \neq 0\}$. This implies that $|\Omega_0| > 0$ and $|u_n(x)| \to \infty$ a.e in Ω. From Assumption (C1) and Lemma 4.3 we deduce,

$$\begin{aligned} k \geq J(u_n) &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u_n|^{p(x)}dx - \left(\frac{p^-}{p^+}\right)^2 \lambda_* |u_n|^{p^-}\right) dx - \int_\Omega E(x,u_n)dx \\ &\geq \frac{1}{p^+}\int_\Omega \left(|\nabla u_n|^{p(x)}dx - \left(\frac{p^-}{p^+}\right)^2 \lambda_* |u_n|^{p(x)}\right) dx - \int_\Omega E(x,u_n)dx \\ &\geq -\int_\Omega E(x,u_n)dx \to +\infty. \end{aligned}$$

This is a contradiction. Hence $J(u)$ is coercive on $W_0^{1,p(x)}$. □

Lemma 4.4. *The functional J is semi-continuous on $W_0^{1,p(x)}$.*

Proof. The functional J_1 is weakly lower semi-continuous. Fix $u \in W_0^{1,p(x)}$ and $\varepsilon > 0$. Since J_1 is convex (condition (B4)), for any $v \in W_0^{1,p(x)}$, we obtain

$$\int_\Omega A(x,\nabla v)dx \geq \int_\Omega A(x,\nabla u)dx + \int_\Omega a(x,\nabla u).(\nabla v - \nabla u)dx.$$

Using assumption (B2) and inequality (2.2) we have

$$\begin{aligned} \int_\Omega A(x,\nabla v)dx &\geq \int_\Omega A(x,\nabla u)dx - \int_\Omega |a(x,\nabla u)| \cdot |\nabla v - \nabla u|\, dx \\ &\geq \int_\Omega A(x,\nabla u)dx - c_1\int_\Omega |\nabla(v-u)|\, dx - c_1\int_\Omega |\nabla u|^{p(x)-1}|\nabla(v-u)|\, dx \\ &\geq \int_\Omega A(x,\nabla u)dx - c_2|1|_{q(x)}|\nabla(v-u)|_{p(x)} - c_3\left||\nabla u|^{p(x)-1}\right|_{q(x)}|\nabla(v-u)|_{p(x)} \\ &\geq \int_\Omega A(x,\nabla u)dx - c_4\,||v-u|| \\ &\geq \int_\Omega A(x,\nabla u)dx - \varepsilon, \end{aligned}$$

where c_2, c_3, c_4 are constant and $||v-u|| < \delta = \varepsilon/4$ and $q(x) = p(x)/(p(x)-1)$. On the other hand let

$$\mathcal{A} = \left\{u \in W_0^{1,p(x)}(\Omega) : u|_{\partial\Omega} = 0\right\}$$

is the class of admissible functions. In view of Lemma 4.3, $J[u]$ is defined and $J[u] \to \infty$ as $||Du|| \to \infty$.
Let

$$m := \inf_{u\in\mathcal{A}}[J[u]. \tag{4.9}$$

Select a sequence $\{u_k\} \in \mathcal{A}(k = 1, 2, \cdots, n)$, so that

$$J[u_k] = \int_\Omega \frac{1}{p(x)}(|\nabla u_k|^{p(x)}dx + \int_\Omega G(x,u_k)dx \to m \quad \text{as} \quad k \to \infty. \tag{4.10}$$

Thus $\{u_k\}_{k=1}^{\infty}$ is a minimizing sequence. Since $1 < p(x) < \infty$ and $L^{p(x)}$ is reflexive, there exists a subsequence $\left(u_{k_j}\right)_{j=1}^{\infty} \subset \{u_k\}_{k=1}^{\infty}$ and a function $u \in W_0^{1,p(x)}$ such that

$$u_{k_j} \rightharpoonup u \quad \text{in} \quad W_0^{1,p(x)} \quad \text{and} \quad u = 0 \quad \text{on} \quad \partial\Omega. \tag{4.11}$$

From the Lemma 4.3, we can deduce from (4.11) an appropriate sub sequence $\left(u_{k_j}\right)_{j=1}^{\infty}$ such that

$$J[u] \leq \lim_{j\to\infty} \inf\left[J[u_{k_j}]\right]. \tag{4.12}$$

Since J_1 is weakly lower semi continuous, then

$$J_1(u) \leq \lim_{k\to\infty} \inf J_1(u_k),$$

and from (4.10)

$$J[u] \leq \lim_{k\to\infty} \inf J(u_k), \tag{4.13}$$

and so

$$J[u] \leq m. \tag{4.14}$$

But from (4.9), we have $m \leq J[u]$ which implies that m is the global minimizer of J. □

5 PROOF OF THE THEOREM 4.1

Let us apply the Mountain pass theorem (Evans (1997), Theorem 2, PP 480) to prove the existence of minimiser. Recall that a functional $J \in C^1(H;\mathbb{R})$ satisfies the Palais-Smale compactness condition (see Evans (1997), pp 477) if each sequence $\{u_k\}_{k=1}^{\infty} \subset H$ such that

i. $\{J[u_k]\}_{k=1}^{\infty}$ is bounded,
ii. $J'[u_k] \to 0 \quad \text{in} \quad H$

has a convergent subsequence in H.

Proof. The conclusion of the theorem follows from the Mountain Pass Theorem, provided that the following two claims are demonstrated.
Claim 1:
Under the assumptions (A4) and (C3), the functional J satisfies the Palais-Smale condition Let $(u_n)_n$ be a Palais-Smale sequence for the functional J, then $J(u_n)$ is bounded and $J'(u_n) \to 0$. We show that $(u_n)_n$ is bounded in $W_0^{1,p(\alpha)}(\Omega)$. Using Assumption (C3), since $J(u_n)$ is bounded, we have

$$\begin{aligned} k_1 &\geq \int_\Omega \frac{1}{p(x)}|\nabla u_n|^{p(x)}dx + \int_\Omega G(x,u_n)dx \\ &\geq \frac{1}{p^+}\underbrace{\int_\Omega |\nabla u_n|^{p(x)}}_{I[\nabla u_n]}dx + \int_\Omega G(x,u_n)dx \\ &\geq \frac{1}{p^+}\left(I[u_n]\right) + \int_\Omega \frac{u_n}{\theta}\varrho(x,u_n)dx + k_2, \end{aligned} \tag{5.1}$$

where k_1, k_2 are constant. Note that

$$\langle J'[u_n], u_n\rangle = \int_\Omega |\nabla u_n|^{p(x)}\, dx + \int_\Omega g(x, u_n)u_n dx. \tag{5.2}$$

This implies that

$$k_1 \geq \left(\frac{1}{p^+} + \frac{1}{\theta}\right) I[\nabla u_n] + \frac{1}{\theta}\langle J'[u_n], u_n\rangle + k_2. \tag{5.3}$$

Suppose now $(u_n)_n$ unbounded in $W_0^{1,p(x)}(\Omega)$; so $||u_n|| \geq 1$ for large value of n. Then

$$||u_n||^{p^-} \leq I[\nabla u_n] \leq ||u_n||^{p^+}$$

for large value of n. Furthermore, $J'[u_n] \to 0$ assures that $\exists k_3 > 0$ such that

$$-k_3\, ||u_n|| \leq \langle J'[u_n], u_n\rangle \leq k_3\, ||u_n||. \tag{5.4}$$

Thus from equations (5.3) and (5.4), we get

$$k_1 \geq \left(\frac{1}{p^+} + \frac{1}{\theta}\right) ||u_n||^{p^-} - \frac{k_3}{\theta}\, |0u_n|0 + k_2. \tag{5.5}$$

Since $p^- > 1$ and $\left(\frac{1}{p^+} + \frac{1}{\theta}\right) > 0$, we have from (5.5),

$$\left(\frac{1}{p^+} + \frac{1}{\theta}\right) ||u_n||^{p^-} - \frac{k_3}{\theta}\, ||u_n|| + k_2 \to \infty \quad \text{as} \quad n \to \infty,$$

which is a contradiction. So $(u_n)_n$ is bounded in $W_0^{1,p(x)}(\Omega)$.

Claim 2:

There exists $r > 0$ and $\alpha > 0$ such that $J(u) \geq \delta$ for all $u \in W^{1,p(x)}(\Omega)$ with $||x|| = r$.

From the assumptions (A4) and (C4) we have

$$G(x, \xi) \leq \varepsilon|\xi|^{p^+} + C(\varepsilon)|\xi|^{q(x)} \;\; \forall (x, \xi) \in \Omega \times \mathbb{R}.$$

For small $||u||$, we have

$$\begin{aligned} Ju &\geq \frac{1}{p^+} I[u] + \int_\Omega G(x, u)dx \\ &\geq \frac{1}{p^+}\, ||u|\,|^{p^+} + \varepsilon \int_\Omega |u|^{p^+} dx + C(\varepsilon) \int_\Omega |u|^{q(x)} dx. \end{aligned}$$

Recall that

$$1 < p^- \leq p(x) \leq p^+ < q^- \leq q(x) \leq q^+ < \infty.$$

Then we have the following embeddings

$$W_0^{1,p(x)}(\Omega) \hookrightarrow L^{p^*}(\Omega), \quad W_0^{1,p(x)}(\Omega) \hookrightarrow L^{q(x)}(\Omega)$$

and as a result with continuous and compact embeddings, there exists constants $k_4, k_5 > 0$ such that

$$||u||_{L_{p^*}} \leq k_4||u||_{p(x)}; \quad ||u||_{x_{(4)}} \leq k_4||u||_{p(x)}, \quad \forall u \in W_0^{1,p(x)}(\Omega).$$

For small $||u||$, we then have

$$\int_\Omega |u|^{q(x)}dx \leq ||u||^{p^-}_{q(x)} \leq k_7||u||^{p^-}_{p(x)}.$$

Thus,

$$\begin{aligned} J[u] &\geq \frac{1}{p^+}\,||u|\,|^{p^+}_{p(x)} + \varepsilon(k_4)^{p^+}|||u||^{p^+}_{p(x)} + k_7||u||^{p^-}_{p(x)} \\ &\geq ||u||^{p^+}_{p(x)}\left(\frac{1}{p^+} + \varepsilon\,(k_4)^{p^+} + k_7||u||^{q-p^+}\right), \end{aligned}$$

where $k_7 = k_6C(\varepsilon)$. Since $q^- - p^+ > 0$, and the function $\xi \to \left(\frac{1}{p^+} + \varepsilon\,(k_4)^{p^+} + k_7||u||^{q^- - p^+}\right)$ is strictly positive near zero. Taking $||u|| = r$ and let

$$\alpha = r^{p^+}\left(\frac{1}{p^+} + \varepsilon\,(k_4)^{p^+} + k_7 r^{q^- - p^+}\right).$$

This implies that $\exists r > 0$ and $\delta > 0$ such that $J[u] \geq \delta, \forall u \in W_0^{1,p(x)}(\Omega)$.
Under assumptions (A4) and (C2), and by following the arguments in the proof of Lemma 3.2 in EL. Amrouss and Kissi (2013) with minor modifications, it follows that the functional $J(\epsilon u)$ has a local maximum at $\epsilon = 0$ for all u. Moreover, $J(\epsilon u) \to -\infty$ as $\epsilon \to +\infty$ for some $u \in W_0^{1,p(x)}(\Omega)$. This implies there exists $h \in W_0^{1,p(x)}(\Omega)$ such that $||h|| = r$ and $J(h) < 0$. By the Mountain Pass Theorem, J has a critical value $\mu \geq \delta$. Let u be the associated critical point. Since $J(u) \geq \delta$ and $J(0) = 0$, u is necessarily a non-zero solution to problem (1.1), which completes the proof. □

6 CONCLUSION

In this paper, we have successfully established the existence of solutions for a specific class of Dirichlet boundary value problems involving the $p(x)$-Laplacian operator within a bounded domain. By employing a variational approach, we demonstrated that the interaction between the variable exponent framework and the non-decreasing continuous function $b(u)$ allows for a robust analytical structure. The application of the Mountain Pass Theorem served as the core of our methodology, enabling us to identify critical points of the associated energy functional under the prescribed growth and regularity conditions on the nonlinearity $f(x, u)$. These results contribute to the broader understanding of non-homogeneous elliptic problems, confirming that the qualitative properties of the variable exponent $p(x)$ and the specific boundary conditions sufficiently support the existence of at least one non-trivial solution.

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