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Original Article

A Unique Fixed-Point Theorem for Self-Mapping in Dislocated Quasi-b-Metric Spaces

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Abstract

The aim of this manuscript is to establish a unique fixed-point result involving a single continuous self-map satisfying certain contractive condition in the frame work of dislocated quasi b-metric space. Furthermore, we prove the existence and uniqueness of fixed point. The present result improves and generalizes several well-known comparable results in the existing literature. We also provide an example in support of our main result.

Keywords: Dislocated quasi-b-metric space, contraction mapping, Cauchy sequence, fixed point

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1. INTRODUCTION

One of the most dynamic research topics in non-linear analysis is fixed point theory. The most important result of this theory is contraction mapping theorem which was proved by the Polish Mathematician Banach (1922) called the Banach contraction mapping principle (Banach, 1922). This principle has been generalized by several authors by putting different type of contractive conditions either on the mappings or on the ambient spaces (Sharma, 2018). The celebrated Banach contraction principle was introduced and generalized by Hitzler & Seda (2000) and Hitzler (2001) in a complete dislocated metric space. Since then,

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the idea of dislocated quasi metric space was introduced by Zeyada et al. (2005) for the first time. The most fascinating property of this space was that self-distance need not to be necessarily zero. Afterwards, the idea of dislocated quasi b-metric space was presented by Rahman & Sarwar (2016).

Recently, some fixed-point result has been proved by Kidane and Aynalem (2019) in the setting of dislocated quasi-b-metric spaces. Inspired by the result of Kidane and Aynalem, the aim of this research work is to establish a unique fixed point theorem for maps in the frame work of dislocated quasi-b-metric spaces by modifying their contractive condition. Our result extends, improves and generalizes comparable results in the existing literature. Moreover, we provided an example in support of our main result.

2. PRELIMINARIES

Note: Throughout this research work R^+ represents the set of non-negative real numbers and N represents the set of natural numbers.

First, we remember some known definitions and lemma.

Definition 2.1 (Rahman & Sarwar, 2016). Let X be a non-empty set. A function $d: X \times X \rightarrow [0, \infty)$ is said to be dislocated quasi b-metric if the following conditions are satisfied; (d_1) $d(x; y) = d(y; x) = 0 \Rightarrow x = y$; (d_2) There exists $k \geq 1$ such that $d(x, y) \leq k[d(x, z) + d(z, y)] \forall\, x, y, z \in X$.

In this case, the pair (X, d) is called dislocated quasi -b -metric spaces or in short $(dq\ b)$ metric spaces.

Definition 2.2 (Rahman & Sarwar, 2016). A sequence $\{x_n\}$ in a dislocated quasi-b-metric space(X, d) is said to converge to a point $x \in X$ if and only if

$$\lim_{n\to\infty} d(x_n, x) = \lim_{n\to\infty} d(x, x_n) = 0.$$

Definition 2.3(Rahman & Sarwar, 2016). Let (X, d) be a dislocated quasi-b-metric space. Then a sequence $\{x_n\}$ is said be a Cauchy sequence if for each $\varepsilon > 0$, there exists $n_0(\varepsilon) \in N$ such that $d(x_n, x_m) < \varepsilon$ for all $n, m > n_0(\varepsilon)$ that is, $\lim_{n,m\to\infty} (x_n, x_m) = 0$.

Definition 2.4(Rahman & Sarwar, 2016). A dislocated quasi-b-metric space (X, d) is called complete if every Cauchy sequence $\{x_n\}$ in (X, d) converges to a point $x \in X$.

Definition 2.5(Rahman & Sarwar, 2016). Let (X, d_1) and (Y, d_2) be two dislocated quasi-b-metric spaces, then the mapping $T: X \rightarrow Y$ is said to be continuous if for each sequence $\{x_n\}$ which is convergent to x_0 in X, the sequence Tx_n converges to Tx_0 in Y.

Lemma 2.1(Rahman & Sarwar, 2016) The Limit of a convergent sequence in dislocated quasi-b-metric space is unique.

Definition 2.6 Let $T: X \to X$ be a self-map in a complete metric space (X, d). Then T is said to be a contraction mapping if there exists a constant $k \in [0, 1)$ such that $d(Tx, Ty) \leq kd(x, y)\ \forall x, y, z \in X$.

Definition 2.7. Let X be a non-empty set and $T: X \to X$ a self-map. We say that x is a fixed point of T if $Tx = x$.

Theorem 2.1(Kidane and Aynalem, 2019). Let (X, d) be a complete dislocated quasi metric space and let $T: X \to X$ be a continuous self-mapping satisfying the following condition:

$$d(Tx, Ty) \leq a_1\, d\,(x, y) + a_2\left[\frac{d(x, Tx)d(y, Ty)}{d\,(x, y)}\right] + a_3[d(x, Tx) + d\,(y, Ty)]$$

$$+a_4[d(x, Ty\,) + d\,(y, Tx)] + a_5[d(x, Tx) + d\,(x, y)] + a_6[d\,(y, Ty)$$

$$+d(x, y)] + a_7[d(x, Ty) + d(x, y)] + a_8[d(y, Tx\,) + d\,(x, y)]$$

$\forall x, y \in X$ with $d(x, y) \neq 0$ where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8 \geq 0$ and

$0 \leq a_1 + a_2 + 2a_3 + 4a_4 + 2a_5 + 2a_6 + 3a_7 + 3a_8 < 1$. Then T has a unique fixed point.

3. RESULT

Theorem 3.1 Let (X, d) be a complete dislocated quasi-b metric space with constant coefficient $k \geq 1$ and let $T: X \to X$ be a continuous self-mapping satisfying the following condition:

$$d(Tx, Ty) \leq a_1\, d\,(x, y) + a_2\left[\frac{d(x, Tx)d(y, Ty)}{d\,(x, y)}\right] + a_3[d(x, Tx) + d\,(y, Ty)]$$

$$+a_4[d(x, Ty\,) + d\,(y, Tx)] + a_5[d(x, Tx) + d\,(x, y)] + a_6[d\,(y, Ty) \qquad (3.1)$$

$$+d(x, y)] + a_7[d(x, Ty) + d(x, y)] + a_8[d(y, Tx\,) + d\,(x, y)]$$

$$+a_9[d\,(x, Ty) + d\,(y, Ty)]$$

$\forall x, y \in X$ with $d(x, y) \neq 0$ where $a_1, a_2, a_3, a_4, a_5, a_6, a_7, a_8,\ a_9 \geq 0$ are non-negative constants and $a_2 + k(a_1 + 2a_5 + a_6) + (k + 1)a_3 + (2k^2 + 2k)a_4 + (k^2 + 2k)(a_7 + a_8) + (k^2 + k + 1)a_9 < 1$. Then T has a unique fixed point.

Proof: Let x_0 be any arbitrary point of X. We can define a sequence $\{x_n\} \in X$ such that $x_1 = Tx_0$, $x_2 = Tx_1, \cdots, x_{n+1} = Tx_n$ for $n = 0, 1, 2, \cdots$.

Replacing x by x_{n-1} and y by x_n in equation(3.1), we have

$$d(x_n, x_{n+1}) = d(Tx_{n-1}, Tx_n) \leq a_1 d(x_{n-1}, x_n) + a_2 \left[\frac{d(x_{n-1}, Tx_{n-1})d(x_n, Tx_n)}{d(x_{n-1}, x_n)}\right]$$
$$+a_3[d(x_{n-1}, Tx_{n-1}) + d(x_n, Tx_n)] + a_4[d(x_{n-1}, Tx_n) + d(x_n, Tx_{n-1})]$$
$$+a_5[d(x_{n-1}, Tx_{n-1}) + d(x_{n-1}, x_n)] + a_6[d(x_n, Tx_n) + d(x_{n-1}, x_n)]$$
$$+a_7[d(x_{n-1}, Tx_n) + d(x_{n-1}, x_n)] + a_8[d(x_n, Tx_{n-1}) + d(x_{n-1}, x_n)]$$
$$+a_9[d(x_{n-1}, Tx_n) + d(x_n, Tx_n)].$$

Based on our definition of the sequences, the above inequality becomes

$$d(x_n, x_{n+1}) = a_1 d(x_{n-1}, x_n) + a_2 \left[\frac{d(x_{n-1}, x_n)d(x_n, x_{n+1})}{d(x_{n-1}, x_n)}\right]$$
$$+a_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1})] + a_4[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)]$$
$$+a_5[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)] + a_6[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)]$$
$$+a_7[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)] + a_8[d(x_n, x_n) + d(x_{n-1}, x_n)]$$
$$+a_9[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})].$$

Simplifying the above inequality, we have

$$d(x_n, x_{n+1}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$
$$+a_4[d(x_{n-1}, x_{n+1}) + d(x_n, x_n)] + a_5[d(x_{n-1}, x_n) + d(x_{n-1}, x_n)]$$
$$+a_6[d(x_n, x_{n+1}) + d(x_{n-1}, x_n)] + a_7[d(x_{n-1}, x_{n+1}) + d(x_{n-1}, x_n)]$$
$$+a_8[d(x_n, x_n) + d(x_{n-1}, x_n)] + a_9[d(x_{n-1}, x_{n+1}) + d(x_n, x_{n+1})].$$

Using the triangle inequality on the 4th, 7th, 8th and 9th terms we have

$$d(x_n, x_{n+1}) \leq a_1 d(x_{n-1}, x_n) + a_2 d(x_n, x_{n+1}) + a_3[d(x_{n-1}, x_n) + d(x_n, x_{n+1})]$$

$$+a_4k[d(x_{n-1},x_n)+d(x_n,x_{n+1})+d\,(x_{n-1},x_n)+d(\,x_n,x_{n+1})]$$

$$+a_5[d(x_{n-1},x_n)+d\,(x_{n-1},x_n)]+a_6[d\,(x_n,x_{n+1})+d(x_{n-1},x_n)]$$

$$+a_7\big[k\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)+d(x_{n-1},x_n)\big]$$

$$+a_8\big[k\big(d(x_{n-1},x_n)+d(x_n,x_{n+1})\big)+d\,(x_{n-1},x_n)\big]$$

$$+a_9[k(d\,(x_{n-1},x_n)+d(x_n,x_{n+1}))+d\,(x_n,x_{n+1})].$$

Simplification yields

$$d(x_n\,,x_{n+1})\leq\frac{a_1\,+a_3+2(ka_4\,+a_5\,)+a_6+(a_7{+}a_8\,)(k+1)+ka_9}{1-(a_2+a_3+a_6\,+k(2a_4\,+a_7\,+a_8\,)+(k+1)a_9\,)}d(x_{n-1},x_n)$$

$$\leq\lambda d(x_{n-1},x_n).$$

Where $\lambda=\frac{a_1\,+a_3+2(ka_4\,+a_5\,)+a_6+(a_7+a_8\,)(k+1)+ka_9}{1-(a_2+a_3+a_6\,+k(2a_4\,+a_7\,+a_8\,)+(k+1)a_9\,)}<\frac{1}{k}$.

Also, we can show that $d(x_{n-1}\,,x_n)\leq\lambda d(x_{n-2},x_{n-1})$.

Proceeding in similar fashion, we get

$d(x_n\,,x_{n+1})\leq\lambda^n d(x_0,x_1)$.

Now, to show that $\{x_n\}$ is a Cauchy sequence, consider for any two non-negative integers m & n with $m>n$ and using(d_2) of definition 2.1, we have

$d(x_n,x_m)\leq k\,d(x_n\,,x_{n+1})+k^2d(x_{n+1}\,,x_{n+2})+\cdots k^{m-n}[d(x_{m-1},x_m)$

$\leq k\lambda^n d(x_0,x_1)+k^2\lambda^{n+1}d(x_0,x_1)+k^{m-n}\lambda^{m-1}d(x_0,x_1)$

$$=k\lambda^n(1+k\lambda+(\mathrm{k}\lambda)^2+\cdots+(\mathrm{k}\lambda)^{m-n-1})d(x_0,x_1)$$

$$\leq\frac{k\lambda^n}{1-k\lambda}d(x_0,x_1)$$

$$=\frac{k\lambda^n}{1-k\lambda}\to 0\text{ as n }\to\infty.$$

This implies that, $d(x_n,x_m)\to 0$ as $n,m\to\infty$.

This shows that $\{x_n\}$ is a Cauchy sequence in a completely dislocated quasi b- metric space. As a result, there must exist a point $x^*\in X$ such that $\lim_{n\to\infty}x_n=x^*$.

Furthermore, since T is continuous, we have

$T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) = \lim_{n\to\infty} x_{n+1} = Tx^*$.

This implies that $Tx^* = \lim_{n\to\infty} x_{n+1}$.

Thus $Tx^* = x^*$. Therefore, x^* is a fixed point of T.

Uniqueness: Now, we show that x^* is a unique fixed point of T.

Suppose there is another fixed point $y^*$of T such that $Ty^* = y^*$for $x^* \neq y^*$.

Consider $d(x^*, y^*) = d(Tx^*, Ty^*)$.

If $x^*$is fixed point of T then $d(x^*, x^*) = 0$. One has to show that if x^* is a fixed point of T then $d(x^*, x^*) = 0$. Similarly, $d(y^*, y^*) = 0$.

So, we have the following inequality

$$d(x^*, y^*) = d(Tx^*, Ty^*)$$

$$\leq a_1\, d\,(x^*, y^*) + a_2\left[\frac{d(x^*, Tx^*)d(y^*, Ty^*)}{d\,(x^*, y^*)}\right] + a_3[d(x^*, Tx^*) + d\,(y^*, Ty^*)]$$

$$+a_4[d(x^*, Ty^*) + d\,(y^*, Tx^*)] + a_5[d(x^*, Tx^*) + d\,(x^*, y^*)] + a_6[d\,(y^*, Ty^*)$$

$$+d(x^*, y^*)] + a_7[d(x^*, Ty^*) + d(x^*, y^*)] + a_8[d(y^*, Tx^*) + d\,(x^*, y^*)]$$

$$+ a_9[d\,(x^*, Ty^*) + d\,(y^*, Ty^*)].$$

$$= a_1\, d\,(x^*, y^*) + a_2\left[\frac{d(x^*, x^*)d(y^*, y^*)}{d\,(x^*, y^*)}\right] + a_3[d(x^*, x^*) + d\,(y^*, y^*)]$$

$$+a_4[d(x^*, y^*) + d\,(y^*, x^*)] + a_5[d(x^*, x^*) + d\,(x^*, y^*)] + a_6[d\,(y^*, y^*)$$

$$+d(x^*, y^*)] + a_7[d(x^*, y^*) + d(x^*, y^*)] + a_8[d(y^*, x^*) + d\,(x^*, y^*)]$$

$$+a_9[d\,(x^*, y^*) + d\,(y^*, y^*)].$$

As $x^*$and y^* are fixed points of T from the above inequality, we have

$d(x^*, x^*) = 0$ and $d(y^*, y^*) = 0$.

Hence, we get

$$d(x^*, y^*) = a_1\, d\,(x^*, y^*) + a_4[d(x^*, y^*) + d\,(y^*, x^*)] + a_5\, d\,(x^*, y^*)$$
$$+a_6 d(x^*, y^*) + 2a_7 d\,(x^*, y^*) + a_8[d(y^*, x^*) + d\,(x^*, y^*)]$$
$$+a_9 d\,(x^*, y^*). \qquad (3.2)$$

$$d(x^*, y^*) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8 + a_9) d\,(x^*, y^*)$$
$$+(a_4 + a_8) d\,(y^*, x^*).$$

Similarly, we have

$$d(y^*, x^*) \leq (a_1 + a_4 + a_5 + a_6 + 2a_7 + a_8 + a_9)d(y^*, x^*) + (a_4 + a_8)d(x^*, y^*). \tag{3.3}$$

Adding equations(3.2) and (3.3), we have

$$[d(x^*, y^*) + d(y^*, x^*)] \leq (a_1 + 2(a_4 + a_7 + a_8) + a_5 + a_6 + a_9)\,[d\,(x^*, y^*) + d\,(y^*, x^*)]. \tag{3.4}$$

Since $(a_1 + 2(a_4 + a_8) + a_5 + a_6 + 2a_7 + a_9) < 1$, hence inequality (3.4) is possible only if $d\,(x^*, y^*) + d\,(y^*, x^*) = 0$.

This implies that $d(x^*, y^*) = d(y^*, x^*) = 0$. Hence $x^* = y^*$.

This is a contradiction to our assumption.

Therefore, by (d_1) of definition 2.1 T has a unique fixed point.

Example 3.1 Let $X = [0,1]$ and $T : X \to X$ is defined as

$$Tx = \frac{x}{4}, \forall\, x \in X$$

with complete dislocated quasi-*b*-metric space given by

$$d(x, y) = |2x - y|^2 + |2x + y|^2, \forall\, x, y \in X.$$

Then (X, d) is dislocated quasi-b-metric space with coefficient $k = 2$, but since $d(1,1) \neq 0$, we get (X, d) is not a quasi-b-metric space.

It is obvious that (X, d) is a dislocated quasi-b- metric space.

$$d(Tx, Ty) = d\left(\frac{x}{4}, \frac{y}{4}\right) = |\tfrac{x}{2} - \tfrac{y}{4}|^2 + |\tfrac{x}{2} + \tfrac{y}{4}|^2.$$

$$d(Tx, Ty) = d\left(\frac{x}{4}, \frac{y}{4}\right) = \frac{1}{16}(|2x - y|^2 + |2x + y|^2)$$

$$\leq \tfrac{1}{4}(|2x - y|^2 + |2x + y|^2) \text{ for all } x, y \in X.$$

Hence, $d(Tx, Ty) \leq \mathrm{k}d(x, y)$ where $K(t) = \frac{1}{4}$, $t \in X$.

In particular, if we take $a_1 = \frac{1}{16}, a_2 = \frac{1}{12}, a_3 = \frac{1}{72}, a_4 = \frac{1}{144}, a_5 = \frac{1}{36}, a_6 = \frac{1}{24}$

$$, a_7 = \frac{1}{27}, a_8 = \frac{1}{54}, a_9 = \frac{1}{49}$$

Since $a_2 + k(a_1 + 2a_5 + a_6) + (k+1)a_3 + (2k^2 + 2k)a_4 + (k^2 + 2k)(a_7 + a_8)$

$$+(k^2 + k + 1)a_9 = \frac{241}{252} < .1$$

All the conditions of Theorem 3.1 are satisfied having $x = 0$ as a unique fixed point of T in X.

Remark: In Theorem 3.1 if $k = 1$ and $a_9 = 0$, then we get Theorem 2.1.

4. DISCUSSION

We are providing an overview of a study conducted by Kidane and Aynalem in 2019. They developed and demonstrated the existence and uniqueness of fixed point outcomes for specific maps within dislocated quasi (dq) metric spaces. In our current study, we have constructed a unique fixed point theorem and demonstrated the existence and uniqueness of a fixed point theorem for maps within the context of complete dislocated quasi-b-metric spaces. This established theorem expands and generalizes previous findings in the literature. Additionally, we have provided an illustration to help illustrate the key finding. It would be helpful to know more specific details or questions we may have regarding this study.

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CONFLICT OF INTEREST

There are no conflicts of interest.

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