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Original article

The Fourier -Brós -Iagolnitzer Transform Characterization of Ultradistributions

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Abstract

Ultradifferentiable function spaces were introduced in order to generalize Schwartz distribution theory. The (Fourier -Brós -Iagolnitzer) transform is known to be the right tool to characterizes microlocal analyticity and microlocal smoothness. In this paper, we characterize microlocal ultradifferentiablity of ultradistrbutions by using a more general class of FBI transforms.

Key words: FBI transform, ultradifferentiable classes, ultradistributions.

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1 Introduction

The classical FBI transform is defined as

$$\mathcal{F}u(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-x')-|\xi||x-x'|^2} u(x')dx', \quad x,\xi \in \mathbb{R}^m \; m \geq 1, \tag{1.1}$$

where u is a continuous function of compact support in $\mathbb{R}^m$ or distribution of compact support in which case the integral is understood in the duality sense. This transform shown to be the right tool to study the real analyticity, microlocal analyticity, microlocal smoothness, microlocal Gevrey regularity and has been used to study the regularity of solutions of linear and nonlinear partial differential equations (see Barostichi and Petronilho (2009); Berhanu (2009); Berhanu and Hounie (2012); Berhanu et al. (2008); Bros and Iagolnitzer (1973); Hörmander (2015); Yesuf and Tadesse (2020)).

In Berhanu and Hounie (2012) a more general class of FBI transforms was introduced including the class

$$\mathcal{F}_u^k(x,\xi) = \int_{\mathbb{R}^m} e^{i\xi.(x-y)-|\xi||x-y|^{2k}} u(y)dy, \; x,\xi \in \mathbb{R}^m, k = 1,2,3,\ldots \tag{1.2}$$

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The authors used these more general transforms to characterize microlocal analyticity and microlocal smoothness. Our goal here to characterize the microlocal regularity of ultradistributions by using more generalize class of FBI transforms.

The organization of this paper is as follows: In section 2 we introduce the basic notions and definitions. Section 3 characterizes microlocal ultradifferentiablity of ultradistributions. Finally, section 4 gives the conclusion of the work.

2 Basic notions and definitions

We now give briefly the definition of ultradifferentiable functions. Spaces of these functions were introduced in order to generalize Schwartz distribution theory. For the reader who is interested to learn more about this sequence and their properties, and the function spaces that follows, we recommend the papers [Adwan and Hoepfner (2010),Adwan and Hoepfner (2015), Komatsu (1973)].

Let $(M_j)_{j\in\mathbb{N}_0}$ be a sequence of positive real numbers with the properties:
(1)(Initial Conditions):

$$M_0 = M_1 = 1. \tag{2.1}$$

(2)(Non-quasianalyticity):

$$\sum_{j=1}^{\infty} \frac{M_j}{M_{j+1}} < \infty, \tag{2.2}$$

which guarantees the existence of non trivial function of compact support.
(3)(Stability under differential operators): There exists a constant $A, H > 0$, independent of j, k, such that for all $j, k \in \mathbb{N}_0$,

$$M_{j+k} \leq AH^{j+k} M_j M_k \tag{2.3}$$

(4)(Invariance under composition): For all $j, k \in \mathbb{N}_0$ with $0 \leq j \leq k$, we have

$$\binom{k}{j} M_{k-j} M_j \leq M_k. \tag{2.4}$$

(5)(Logarithmic convexity): For all $j \in \mathbb{N}_0$,

$$M_j^2 \leq M_{j-1} M_{j+1}, \tag{2.5}$$

and this in turn implies that for all $j, k \in \mathbb{N}_0$,

$$M_j M_k \leq M_{j+k}. \tag{2.6}$$

(7)(Faá di Bruno): For all $j, k \in \mathbb{N}$, if $n = jk$, there is a constant $C > 1$, independent of n, so that

$$M_j^k \leq C^n M_{n-k}. \tag{2.7}$$

This condition is used to apply multi-variable Faá di Bruno formula for computing the derivatives of compositions of functions.

Definition 2.1. Let $\Omega \subset \mathbb{R}^m$ be an open subset and $(M_j)_{j\in\mathbb{N}_0}$ be as above. The ultradifferentiable (Denjoy-Carleman) spaces, $\mathcal{E}^M(\Omega)$, is defined as the set of all functions f in $C^\infty(\Omega)$ that satisfies the following property: for each $K \subset\subset \Omega$ there exist a constant $C > 0$, depending on K and f, such that

$$|\partial^\alpha f(x)| \leq C^{|\alpha|+1} M_{|\alpha|}, \quad \forall \alpha \in \mathbb{N}_0^m, \ \forall x \in K. \tag{2.8}$$

Example 2.2. Let $s > 1$ be a real number and choose $M_j = (j!)^s$. Then $M = (M_j)$ satisfies (2.1)-(2.7) and $\mathcal{E}^M(\Omega) = G^s(\Omega)$ denotes the $s-$ Gevrey space. If $M_j = j!$, then $M = (M_j)$ satisfies all conditions except (2.2), and $\mathcal{E}^M(\Omega) = C^\omega(\Omega)$ (the space of real analytic functions).

Definition 2.3. For each sequence $M = (M_j)$ of positive numbers we define its associate function $M(t)$ on $[0, \infty)$ by

$$M(t) = \sup_j \log \frac{t^j}{M_j},\ t \in (0, \infty), M(0) = 0.$$

From the definition it follows that for all $t > 0$,

$$\log t \leq M(t) \leq t. \tag{2.9}$$

Note that if $M_j = j!^s$, then $M(t)$ is equivalent to $t^{\frac{1}{s}}$, in the sense that for some constants $h_1 > 0$, $h_2 > 0$, $B_1 > 0, B_2 > 0$ we have

$$M(h_1 t) + \log B_1 \leq t^{\frac{1}{s}} \leq M(h_2 t) + \log B_2 \ \text{ (see Komatsu (1973)).}$$

Lemma 2.4. *Let $M = (M_j)$ be a sequence of positive numbers defined above and $M(t)$ be its associated function. It follows that*
a) For each $c > 0$ there exist $c' > 0$ (depending on the sequence M and c) such that

$$cM(t) \geq M(c't)\ \forall t > 0. \tag{2.10}$$

b) For each $L > 0$ and $k, r \in \mathbb{N}$ such that $k \geq r \geq 0$, we have

$$t^r M_{k-r} \leq \sqrt{A} \frac{H^r}{L^r} M_k e^{\frac{1}{2}M(Lt)},\ t > 0, \tag{2.11}$$

where A and H were defined by (2.3).
c) If $c, \gamma > 0$ and $r \in \mathbb{Q}_+$ (the set of positive rational numbers) then

$$\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi < \infty \tag{2.12}$$

Proof: (a)and (b) follow from [Hoepfner and Medrado (2018), Lemma A.1]. For (c), let $r \in \mathbb{Q}_+$. Then we have

$$\begin{aligned}
\int_{\mathbb{R}^m} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi &= \int_{|\xi|\leq 1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi \\
&\leq \int_{|\xi|\leq 1} e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^r e^{-cM(\gamma|\xi|)} d\xi \\
&\leq \int_{|\xi|\leq 1} e^{-cM(\gamma|\xi|)} d\xi + \int_{|\xi|>1} |\xi|^m e^{-cM(\gamma|\xi|)} d\xi,
\end{aligned}$$

where m is a positive integer greater than r. Then clearly the first term is finite and the second is finite by Lemma A.1 (i) in Hoepfner and Medrado (2018). Thus the whole integral is finite as desired.

Definition 2.5. Let $\Omega \subset \mathbb{R}^m$ be an open set. We shall denote by $D^M(\Omega)$ the vector space of all $\phi \in \mathcal{E}^M(\Omega)$ with compact support in Ω. The space $D^{M'}(\Omega)$ of $M-$ultradistributions is defined to be the dual of $D^M(\Omega)$; that is, $D^{M'}(\Omega)$ is the

space of all linear form u on $D^M(\Omega)$ such that for each $K \subset\subset \Omega$ and for all $\epsilon > 0$ there is a constant $C_\epsilon > 0$ such that

$$|u(\phi)| \leq C_\epsilon \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon^{|\alpha|}}{M_{|\alpha|}} \sup_{x \in K} |\partial^\alpha \phi(x)| \right\}, \ \ \forall \phi \in D^M(K) = D^M(\Omega) \cap C_0^\infty(K).$$

Finally, we define $\mathcal{E}^{M'}(\Omega)$ to be the space of all linear form u on $\mathcal{E}^M(\Omega)$ such that there exist $K \subset\subset \Omega$ and, for all $\epsilon > 0$, a constant $C_\epsilon > 0$ such that

$$|u(\phi)| \leq C_\epsilon \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon^{|\alpha|}}{M_{|\alpha|}} \sup_{x \in K} |\partial^\alpha \phi(x)| \right\}, \ \forall \phi \in \mathcal{E}^M(\Omega).$$

3 Characterization of microlocal ultradifferebntiablity of ultradistributions

Following Berhanu and Hounie (2012), for $\psi \in \mathcal{E}^M(\mathbb{R}^m)$ such that $0 \neq \int |\psi(x)| dx < \infty$, we define the FBI transform of a distribution $u \in \mathcal{E}^{M'}(\Omega)$ with generating function ψ and parameter λ, $0 < \lambda < 1$, by

$$\mathcal{F}_{\psi,\lambda} u(x,\xi) = \langle u(x'), e^{i\xi.(x-x')} \psi\Big(|\xi|^\lambda (x - x') \Big) \rangle,$$

where the right hand side of the equation is understood in duality sense. Note that if $\psi(x) = e^{-|x|^2}$, then $\mathcal{F}_{\psi,\frac{1}{2}}$ is the Fourier-Bros-Iagolnitzer (FBI) transform [see Berhanu and Hounie (2012)].

Let $p(x) = \sum_{|\alpha|=2k} a_\alpha x^\alpha$, where $x \in \mathbb{R}^m$, $a_\alpha > 0$ ($\alpha \in \mathbb{N}_0^m, |\alpha| \leq 2k$) be a positive elliptic homogeneous polynomial of degree $2k$, that is, it satisfies

$$c|x|^{2k} \leq p(x) \leq b|x|^{2k} \ \text{ for some constants } 0 < c \leq b.$$

Take $\psi(x) = e^{-p(x)}$ as a generating function and $\lambda = \frac{1}{2k}$ as a parameter. Then for $u \in \mathcal{E}^{M'}$ we define the FBI transform as

$$\mathcal{F}u(x,\xi) = c_p \langle u(x'), e^{i\xi.(x-x')} e^{|\xi| p(x-x')} \rangle, \ \ x, \xi \in \mathbb{R}^m.$$

Let $\chi(x) \in D^M(\mathbb{R}^m)$, such that $\chi \geq 0$ and $\int_{\mathbb{R}^m} \chi(x) dx = 1$. Set $\sigma(\xi) = \frac{\tilde{\chi}(\xi)}{(2\pi)^m}$. Let $u \in \mathcal{E}^{M'}(\mathbb{R}^m)$ then the inversion formula becomes

$$u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(x-x')} \sigma(\epsilon\xi) \mathcal{F}u(x',\xi) |\xi|^{\frac{m}{2k}} dx' d\xi.$$

The validity of this inversion formula follows from [Hoepfner and Medrado (2018), Lemma 3.2].

In the following lemma, U is an open subset of $\mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ containing the origin, $x = (x_1, \ldots, x_m) \in \mathbb{R}^m$, $x' = (x'_1, \ldots, x'_m) \in \mathbb{R}^m$.

Lemma 3.1. *Let $U \subset \mathbb{R}^{2m} = \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ be an open set. Let*

$$Q(x,x',\xi) = i\xi.(x - x') - |\xi| p(x - x').$$

Then for each compact subset $K \subset U$, there exist $C > 0$ such that for all $\alpha \in \mathbb{N}_0^m$, $L > 0$ and $|\xi| > 1$,

$$|\partial_{x'}^{\alpha} e^{Q(x,x',\xi)}| \leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|+1} \text{ on } K.$$

Proof. In what follows, we may increase the constant C from line to line, a finite number of times. Fix $K \subset U$ compact. Applying Faá di Bruno formula (see Constantine and Savits (1996)):

$$\partial_{x'}^{\alpha} e^{Q(x,x',\xi)} = \sum_{r=1}^{|\alpha|} e^{Q(x,x',\xi)} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial_{x'}^{l_j} Q)^{k_j}}{k_j!(l_j!)^{k_j}}, \text{ where}$$

$$\begin{aligned} &p(\alpha, r) = \{(k_1, \ldots, k_{|\alpha|}; l_1, \ldots, l_{|\alpha|}) \text{ for some } 1 \leq s \leq |\alpha|, k_i = 0 \text{ and } l_i = 0 \\ &\text{for } 1 \leq i \leq |\alpha| - s; k_i > 0 \text{ for } |\alpha| - s + 1 \leq i \leq |\alpha|; \text{ and} \\ &0 \prec \alpha_{|\alpha|-s+1} \prec \ldots \prec l_{|\alpha|} \text{ are such that} \\ &\qquad \sum_{i=1}^{|\alpha|} k_i = \lambda, \ \sum_{i=1}^{|\alpha|} k_i |l_i| = \alpha\}. \end{aligned}$$

Thus using the fact that $p \in \mathcal{E}^M(\mathbb{R}^m)$, on K and for $|\xi| > 1$, we have

$$\begin{aligned} |\partial_{x'}^{\alpha} e^{Q(x,x',\xi)}| &= \left| \sum_{r=1}^{|\alpha|} e^{Q(x.x',\xi)} \sum_{p(\alpha,r)} \alpha! \prod_{j=1}^{|\alpha|} \frac{(\partial_{x'}^{\alpha_j} Q(x,x',\xi))^{k_j}}{k_j!(l_j!)^{k_j}} \right| \\ &\leq \sum_{r=1}^{|\alpha|} e^{\Re Q(x.x',\xi)} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} |\frac{(\partial_{x'}^{l_j} Q(x,x',\xi))^{k_j}}{k_j!(l_j!)^{k_j}}| \\ &\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left[|\xi||\partial_{x'}^{\alpha_j}(x - x')| + |\xi||\partial_{x'}^{l_j} p(x - x')| \right]^{k_j}}{k_j!(l_j!)^{k_j}} \\ &\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\xi| C^{|l_j|+1} M_{|\alpha_j|} \right)^{k_j}}{k_j!(l_j!)^{k_j}}. \end{aligned}$$

Now using (2.7) and (2.6), respectively, we obtain

$$\prod_{j=1}^{|\alpha|} M_{|l_j|}^{k_j} \leq \prod_{j=1}^{|\alpha|} C^{k_j|l_j|} M_{|l_j|k_j - k_j} \leq C^{|\alpha|} M_{|\alpha|-r}.$$

Also we have that(see Adwan and Hoepfner (2010), page 2275)

$$\prod_{j=1}^{|\alpha|} \left(\frac{|\xi| C^{|\alpha_j|+1}}{\alpha_j!} \right)^{k_j} \leq \frac{|\xi|^r C^{|\alpha|+r} 4^{|\alpha|}}{|\alpha|!}.$$

Therefore, using $t^r M_{|\alpha|-r} \leq \sqrt{A}(\frac{H}{L})^r M_{|\alpha|} e^{\frac{1}{2}M(Lt)}, t > 0$ (Lemma 2.4, (2.11)), we have

$$
\begin{aligned}
|\partial_{x'}^{\alpha} e^{Q(x,x',\xi)}| &\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\alpha|! \prod_{j=1}^{|\alpha|} \frac{\left(|\xi| C^{|\alpha_j|+1} M_{|\alpha_j|}\right)^{k_j}}{k_j!(\alpha_j!)^{k_j}} \\
&\leq \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} |\xi|^r C^{2|\alpha|+r} 4^{|\alpha|} M_{|\alpha|-r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r C^{|\alpha|+r} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} \sum_{p(\alpha,r)} (\frac{H}{L})^r \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&= e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \frac{(\frac{H}{L})^r}{r!} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!}
\end{aligned}
$$

$$
\begin{aligned}
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} \sum_{r=1}^{|\alpha|} r! \sum_{p(\alpha,r)} \prod_{j=1}^{|\alpha|} \frac{1}{k_j!} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|} C^{|\alpha|} \text{ (see Constantine and Savits (1996), page 515)} \\
&\leq e^{\frac{1}{2}M(L|\xi|)} M_{|\alpha|} e^{\frac{H}{L}} C^{|\alpha|+1}.
\end{aligned}
$$

□

Theorem 3.2. *Let Ω be an open subset of $\mathbb{R}^m$, $u \in \mathcal{E}^{M'}(\Omega)$. Then for each $K \subset\subset \Omega$ and for every $L > 0$, there exist $C_L > 0$ such that*

$$|\mathcal{F}u(x,\xi)| \leq C_L e^{M(L|\xi|)},\ x \in K,\ \xi \in \mathbb{R}^m.$$

Proof. Let $u \in \mathcal{E}^{M'}(\Omega)$. Then by definition $\mathcal{F}u(x,\xi) = \langle u(x'), \varphi(x,x',\xi)\rangle$, where $\varphi(x,x',\xi) = e^{i\xi.(x-x')-|\xi|p(x-x')}$. Let $\Omega \times \Omega_1 \subset \mathbb{R}^m_x \times \mathbb{R}^m_{x'}$ be an open set. Let $K \subset\subset \Omega$, $K_1 \subset\subset \Omega_1$ so that $K' = K \times K_1 \subset\subset \Omega \times \Omega_1$. Now from Definition 2.5, for any $\epsilon_1 > 0$ and suitable $C_1 > 0$ depending on ϵ_1 we have, for $x \in K$,

$$
\begin{aligned}
|\mathcal{F}u(x,\xi)| &= |\langle u(x'), \varphi(x,x',\xi)\rangle| \\
&\leq C_1 \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x' \in K_1} |\partial_{x'}^{\alpha} \varphi(x,x',\xi)| \right\} \\
&= C_1 \sup_{\alpha \in \mathbb{N}_0^m} \left\{ \frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}} \sup_{x' \in K_1} \left| \partial_{x'}^{\alpha} \left(e^{i\xi.(x-x')-|\xi|p_1(x-x')} \right) \right| \right\}.
\end{aligned}
$$

By Lemma 3.1, on such a compact set K' there exist $C > 0$ such that for $\alpha \in \mathbb{N}_0^m$, $L > 0$ and $|\xi| > 1$,

$$\left|\partial_{x'}^{\alpha}\left(e^{i\xi.(x-x')-|\xi|p(x-x')}\right)\right| \leq e^{\frac{1}{2}M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}.$$

Thus for $|\xi| > 1$,

$$\begin{aligned}\sup_{K'}\left|\partial_{x'}^{\alpha}\left(e^{i\xi.(x-x')-|\xi|p(x-x')}\right)\right| &\leq e^{\frac{1}{2}M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}\\ &\leq e^{M(L|\xi|)}e^{\frac{H}{L}}C^{|\alpha|+1}M_{|\alpha|}.\end{aligned}$$

Thus, for $x \in K$,

$$\begin{aligned}|\mathcal{F}u(x,\xi)| &\leq C_1 \sup_{\alpha\in\mathbb{N}_0^m}\left\{\frac{\epsilon_1^{|\alpha|}}{M_{|\alpha|}}C^{|\alpha|+1}e^{M(L|\xi|)}M_{|\alpha|}e^{\frac{H}{L}}\right\}\\ &= C_1Ce^{\frac{H}{L}}e^{M(L|\xi|)}\sup_{\alpha\in\mathbb{N}_0^m}\left\{(C\epsilon_1)^{|\alpha|}\right\},\ |\xi|\geq 1\\ &\leq C_Le^{M(L|\xi|)},\end{aligned}$$

where we choose $\epsilon_1 < \frac{1}{C}$. For $|\xi| \leq 1$, the estimate holds by continuity. □

Let $M(t)$ be the associated function to the sequence (M_j). The Young conjugate of the associated function (see Petzsche and Vogt (1984)) $w^* : [0,+\infty) \to [0,+\infty]$ is defined by

$$w^*(r) = \sup_{t\geq 0}\{M(t) - rt\}.$$

Another important function is $M^*(s)$, which is defined as the associated function of the sequence $\frac{M_j}{j!}$. Note also that if $M_j = j!^s$ $(s > 1)$, then $M^*(t)$ is equivalent to $\frac{1}{t^{\frac{1}{s-1}}}$ (see Komatsu (1973)).

Let V be a neighborhood of $0 \in \mathbb{R}^m$, $\Gamma \subset \mathbb{R}^n$ be an open acute cone with vertex at the origin, $\delta > 0, \Gamma_\delta = \Gamma \cap \{v : |v| < \delta\}$, $f \in C^0(V \times \Gamma)$. Let $L_j = \partial_{\overline{z}_j}$, $j = 1,\ldots,m$ and $\mathcal{V} = span\{L_1,\ldots,L_m\}$ be $M-$ locally integrable structure of class $\mathcal{E}^M(\Omega)$. Suppose
(1) $L_jf \in L^\infty(V \times \Gamma)$, $1 \leq j \leq m$;
(2) f increases M^*-exponentially, that is, for all $\varrho > 0$ we have

$$|f(x,t)|e^{-\varrho M^*(|t|/\varrho)} < \infty. \tag{3.1}$$

Then from Theorem 3.1, Adwan and Hoepfner (2015) there exist $bf \in D^{M'}(V)$ such that

$$\langle bf, \varphi\rangle = \lim_{\Gamma\ni t\to 0}\int f(x,t)\varphi(x)dx,\ \ \forall\varphi \in D^M(V).$$

Definition 3.3. Let $u \in D^{M'}(\Omega)$, $x_0 \in \Omega$ and $\xi^0 \in \mathbb{R}^m \setminus \{0\}$. We say that u is $M-$microregular at (x_0,ξ^0) if there exist a neighborhood V of x_0, acute open cones $\Gamma_1,\ldots,\Gamma_n \subset \mathbb{R}^m\setminus\{0\}$ and $f_j \in \mathcal{E}^M(V + i\Gamma_j^\delta)$ (for some $\delta > 0$) that increases M^*-exponentially such that
i) $\xi^0.\Gamma_j < 0\ \forall j$,

ii) $u = \sum_{j=1}^{n} bf_j$ near x_0,
iii) $\left|\frac{\partial f_j}{\partial \overline{z}_k}(x,y)\right| \leq C^{N+1}\frac{M_N}{N!}|y|^N \ \forall j = 1,2,\ldots,n; \forall k = 1,2,\ldots m, \ \forall N \in \mathbb{N}.$

The following theorem characterizes the microlocal ultradifferentiablity of ultradistributions. When $M_j = j!^s$ $(s > 1)$, it gives the Gevrey case.

Theorem 3.4. *Let $u \in \mathcal{E}^{M'}(\mathbb{R}^m), x_0 \in \mathbb{R}^m, \xi^0 \in \mathbb{R}^m$ with $|\xi^0| = 1$. Then u is $M-$ microregular at (x_0, ξ^0) if and only if there exist a neighborhood V of x_0, a conic neighborhood Γ of ξ^0 and constants $a, b > 0$ such that for some $\phi \in D^M(\mathbb{R}^m), \phi \equiv 1$ near x_0,*

$$|\mathcal{F}(\phi u)(x,\xi)| \leq ae^{-M(b|\xi|)}, \ \ (x,\xi) \in V \times \Gamma.$$

Proof. We may assume that $x_0 = 0$. Suppose u is $M-$ microregular at (x_0, ξ^0). Then by definition there exist $f_j \in \mathcal{E}^M$ in some truncated wedge $V + i\Gamma_j^\delta$ $(\delta > 0)$, V a neighborhood of 0 and Γ an open cone such that
i) $u = \sum_{j=1}^{n} bf_j$ on V, ii) $\xi^0.\Gamma_j < 0$ and iii)$|\frac{\partial f_j}{\partial \overline{z}_k}| \leq C^{N+1}\frac{M_N}{N!}|y|^N \ \forall k = 1,2,\ldots m \ , j = 1,\ldots,m; \ \forall N \in \mathbb{N}$ with $|f(x+iy)|e^{-\varrho M^*(\frac{|y|}{\varrho})} < \infty \ \forall \varrho > 0$.
Without loss of generality we may assume that $u = bf$. Let $r > 0$ such that $B_{2r} \subset\subset V$ and $\phi \in D^M(B_{2r})$ such that $\phi_{|B_r} \equiv 1$. Let $v \in \Gamma_\delta$ and

$$Q(x,\xi,x') = i\xi.(x - x') - |\xi|p(x - x').$$

Then $\mathcal{F}(\phi u)(x,\xi) = c_p \lim_{t\to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x')}\phi(x')f(x' + itv)dx'$.
Since $\phi \in D^M(\mathbb{R}^m)$, it has an almost holomorphic extension $\tilde{\phi}(x+iy)$ on $V + i\mathbb{R}^m$ with $x-$ support in B_{2r}. Then

$$\mathcal{F}(\phi u)(x,\xi) = c_p \lim_{\lambda\to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x'+i\lambda v)}\tilde{\phi}(x' + i\lambda v)f(x' + i\lambda v)dx'.$$

For $0 < t < 1$, let $D_t = \{x' + i\lambda v \in \mathbb{C}^m : x' \in B_{2r}, t \leq \lambda \leq 1\}$.
Consider the $m-$ form $\omega(z) = e^{Q(x,\xi,z)}\tilde{\phi}(z)f(z)dz$, where $dz = dz_1 \wedge \ldots \wedge dz_m, \ z = x' + iy'$.
Since $\tilde{\phi}(z) = 0$ for $|x'| \geq 2r$ and since $e^{Q(x,\xi,z)}$ is holomorphic in z, by Stokes theorem (see Range (1986), Theorem 1.7) we have

$$\begin{aligned}
\mathcal{F}(\phi u)(x,\xi) &= c_p \lim_{t\to 0^+} \int_{B_{2r}} e^{Q(x,\xi,x'+itv)}\tilde{\phi}(x' + itv)f(x' + itv)dx' \\
&= c_p \int_{B_{2r}} e^{Q(x,\xi,x'+iv)}\tilde{\phi}(x' + iv)f(x' + iv)dx' \\
&+ c_p \lim_{t\to 0^+} \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x,\xi,x'+i\lambda v)}\tilde{\phi}(x' + i\lambda v)\frac{\partial f}{\partial \overline{z}_j}(x' + i\lambda v)d\overline{z}_j \wedge dz \\
&+ c_p \lim_{t\to 0^+} \sum_{j=1}^{m} \int\int_{D_t} e^{Q(x,\xi,x'+i\lambda v)}f(x' + i\lambda v)\frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x' + i\lambda v)d\overline{z}_j \wedge dz \\
&= c_p[I_0(x,\xi) + \lim_{t\to 0^+}\Big(I_1^t(x,\xi) + I_2^t(x,\xi)\Big)].
\end{aligned}$$

Since $v \in \Gamma$ and $\xi^0.\Gamma < 0$, there is a conic neighborhood Γ_1 of ξ^0 and a constant $c > 0$ such that $\xi.v \leq -c|\xi||v|,\ \forall \xi \in \Gamma_1$.
Consider $I_0(x,\xi)$:

$$|I_0(x,\xi)| \leq \sup_{x' \in \overline{B_{2r}}} |\tilde{\phi}(x'+iv)f(x'+iv)| \int_{B_{2r}} e^{\Re Q(x,\xi,x'+iv)} dx'.$$

For $\xi \in \Gamma_1, |\xi| \geq 1$,

$$\begin{aligned}
\Re Q(x,\xi,x'+iv) &= \Re\Big(i\xi.(x-x'-iv) - |\xi|p(x-x'-iv)\Big) \\
&= \xi.v - |\xi|\Re p(x-x'-iv) \\
&= \xi.v - |\xi|p(x-x') + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| - c|x-x'|^2 + O(|v|^2)|\xi| \\
&\leq -c|v||\xi| + O(|v|^2)|\xi|.
\end{aligned}$$

Choosing $|v|$ small such that $O(|v|^2) \leq \frac{c|v|}{2} = c'$, we then have

$$\Re Q(x,\xi,x'+iv) \leq -c'|\xi|,\ \xi \in \Gamma_1, x \in \mathbb{R}^m.$$

Thus, for $\xi \in \Gamma_1, |\xi| \geq 1$, we have $|I_0(x,\xi)| \leq c'' e^{-c'|\xi|} \leq c'' e^{-M(c'|\xi|)}$ for some $c'' > 0$, because $M(t) \leq t$. Since $\frac{I_0(x,\xi)}{e^{-M(c'|\xi|)}}$ is bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$, there are constants $A_0, B_0 > 0$ such that

$$|I_0(x,\xi)| \leq A_0 e^{-M(B_0|\xi|)},\ \forall \xi \in \Gamma_1, |x| < 2r. \tag{3.2}$$

Consider $I_1^t(x,\xi) = \sum_{j=1}^m \int\int_{D_t} e^{Q(x,\xi,x'+i\lambda v)} \tilde{\phi}(x'+i\lambda v) \frac{\partial f}{\partial \overline{z}_j}(x'+i\lambda v) d\overline{z}_j \wedge dz$.
Now

$$\begin{aligned}
\Re Q(x,\xi,x'+i\lambda v) &\leq -c\lambda|v||\xi| - c|x-x'|^{2k}|\xi| + O(|\lambda v|^2)|\xi| \\
&\leq -c\lambda|v||\xi| - c|x-x'|^{2k}|\xi| + A''\lambda^2|v|^2|\xi| \\
&\leq -c\lambda|v||\xi| + A''\lambda|v|^2|\xi| \\
&\leq -c'\lambda|v||\xi| \ \text{ (taking } |v| \text{ small such that } A''|v|^2 \leq \frac{c|v|}{2} = c'|v|).
\end{aligned}$$

Then for $\xi \in \Gamma_1, |\xi| \geq 1$, letting $C' = \sup_{(x',\lambda) \in \overline{B_{2r}} \times [0,1]} |\tilde{\phi}(x'+i\lambda v)|$, we obtain

$$\begin{aligned}
|e^{Q(x,\xi,x'+i\lambda v)} \tilde{\phi}(x'+i\lambda v) \frac{\partial f}{\partial \overline{z}_j}(x'+i\lambda v)| &\leq C' e^{\Re Q(x,\xi,x'+i\lambda v)} |\frac{\partial f}{\partial \overline{z}_j}(x'+i\lambda v)|, \\
&\leq A' e^{-c'\lambda|v||\xi|} e^{-M^*(c\lambda|v|)} \\
&= A' e^{-[M^*(c\lambda|v|) + c\lambda|v|\frac{c'}{c}|\xi|]} \\
&\leq A' e^{-\inf_s\{M^*(s) + s\frac{c'}{c}|\xi|\}} \\
&\leq A' e^{-M(c''|\xi|)},
\end{aligned}$$

since $M(t) \leq \inf_s\{M^*(s) + st\}\ \forall t \geq 0$.
Thus

$$\lim_{t\to 0^+} |I_1^t(x,\xi)| \leq A' e^{-M(c''|\xi|)} \sum_{j=1}^{m} \int_0^1 \int_{B_{2r}} d\overline{z}_j \wedge dz_j \leq a_1 e^{-M(c''|\xi|)}.$$

Therefore, $\lim_{t\to 0^+} |I_1^t(x,\xi)| \leq a_1 e^{-M(b_1|\xi|)},\ \forall \xi \in \Gamma_1, |\xi| \geq 1, x \in B_{2r}$ for some $a_1, b_1 > 0$ independent of t.
But $\frac{|I_1^t(x,\xi)|}{e^{-M(b_1|\xi|)}}$ is uniformly bounded on $\overline{B_{2r}} \times \{\xi : |\xi| \leq 1\}$. Thus, there are $A_1, B_1 > 0$ such that

$$\lim_{t\to 0^+} |I_1^t(x,\xi)| \leq A_1 e^{-M(B_1|\xi|)},\ \forall \xi \in \Gamma_1, |x| < B_{2r}. \tag{3.3}$$

Consider

$$I_2^t(x,\xi) = \sum_{j=1}^{m} \int \int_{D_t} e^{Q(x,\xi,x'+i\lambda v)} f(x'+i\lambda v) \frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x'+i\lambda v) d\overline{z}_j \wedge dz.$$

For $\xi \in \Gamma_1, |\xi| \geq 1$,

$$\begin{aligned} \Re Q(x,\xi,x'+i\lambda v) &\leq -c\lambda|v||\xi| + O(\lambda^2|v|^2)|\xi| - c|\xi||x-x'|^{2k} \\ &\leq O(|v|^2)|\xi| - c|\xi||x-x'|^{2k} \text{ since } \lambda \leq 1 \\ &\leq a'|v|^2|\xi| - c|\xi||x-x'|^{2k}. \end{aligned}$$

Since $\frac{\partial \tilde{\phi}}{\partial \overline{z}_j} \equiv 0$ for $|x'| \leq r$, the integral over $|x'| \leq r$ is zero.
Then for $|x| < \frac{r}{2}$ and $|x'| \geq r$, we have $\Re Q(x,\xi,x'+i\lambda v) \leq a'|v|^2|\xi| - c\frac{r^{2k}}{2^{2k}}|\xi|$.
Choosing $|v|$ small such that $a'|v|^2 \leq c\frac{r^{2k}}{2^{2k+1}} = c''$, we obtain

$$\Re Q(x,\xi,x'+i\lambda v) \leq -c''|\xi|,\ \xi \in \Gamma_1, |\xi| \geq 1.$$

Since f increases M^*-exponentially, for all $\varrho > 0$, there is a constant $d > 0$ such that

$$|f(x'+i\lambda v)| \leq d e^{\varrho M^*(\frac{\lambda|v|}{\varrho})}.$$

Since $\tilde{\phi}$ is almost holomorphic, for each compact set $K \subset B_{2r}(0)$, there is $C_K = C > 0$ such that

$$|\frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x+i\lambda v)| \leq C^{N+1} \frac{M_N}{N!} \lambda^N |v|^N\ \forall N = 1, 2, \ldots.$$

Equivalently, $|\frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x+i\lambda v)| \leq C e^{-M^*(C\lambda|v|)}$. Now taking $C > 1$ and $\varrho = \frac{1}{C}$, we have that

$$\begin{aligned} |f(x'+i\lambda v)||\frac{\partial \tilde{\phi}}{\partial \overline{z}_j}(x+i\lambda v)| &\leq dC e^{\varrho M^*(\frac{\lambda|v|}{\varrho}) - \frac{1}{C}M^*(c\lambda|v|)} \\ &\leq \frac{d}{\varrho} e^{-\varrho M^*(\frac{1}{\varrho}\lambda|v|) + \varrho M^*(\frac{1}{\varrho}\lambda|v|)} = C' \end{aligned}$$

Thus we can get $A_2, B_2 > 0$ independent of t such that

$$\lim_{t\to 0^+} |I_2^t(x,\xi)| \leq A_2 e^{-M(B_2|\xi|)},\ \forall \xi \in \Gamma_1, |x| < \frac{r}{2}. \tag{3.4}$$

Therefore, from (3.2), (3.3) and (3.4), we can find constants $a, b > 0$ such that

$$|\mathcal{F}\phi u(x,\xi)| \leq ae^{-M(b|\xi|)},\ \forall (x,\xi) \in B_{\frac{r}{2}} \times \Gamma_1,$$

where Γ_1 is a conic neighborhood of ξ^0.
Conversely, suppose $|\mathcal{F}(\phi u)(x,\xi)| \leq ae^{-M(b|\xi|)},\ \forall (x,\xi) \in V \times \Gamma$, where V is some neighborhood of $0, \Gamma$ a conic neighborhood of ξ^0 and, $A, B > 0$ are some constants and $\phi \in D^M(\mathbb{R}^m), \phi \equiv 1$ near 0. We will show that u is $M-$microregular at $(0,\xi^0)$. Let $\sigma(\xi) = e^{-|\xi|^2}$. We apply the inversion formula

$$\phi(x)u(x) = \lim_{\epsilon \to 0^+} \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(x-x') - \epsilon^2|\xi^2|} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Let $u_\epsilon(z) = \int_{\mathbb{R}^m \times \mathbb{R}^m} e^{i\xi.(z-x') - \epsilon^2|\xi^2|} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi, \quad z = x + iy \in \mathbb{C}^m$. Clearly $u_\epsilon(z)$ is an entire function of z for each $\epsilon > 0$. For some $a > 0$, write $u_\epsilon(z)$ as

$$u_\epsilon(z) = u_0^\epsilon(z) + u_1^\epsilon(z), \text{ where}$$

$$u_0^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|x'| \leq a} e^{i\xi.(z-x')} \sigma(\epsilon\xi) \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi \text{ and}$$

$$u_1^\epsilon(z) = \int_{\mathbb{R}^m} \int_{|x'| \geq a} e^{i\xi.(z-x')} \sigma(\epsilon\xi) \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Consider $u_0^\epsilon(z)$: Choose $a > 0$ such that $\{x' : |x'| \leq a\} \subset V$. Let $\mathcal{C}_0 = \Gamma, \mathcal{C}_j,\ 1 \leq j \leq n$ be open acute cones (we may take Γ to be acute) such that

$$\mathbb{R}^m = \cup_{j=0}^n \overline{\mathcal{C}_j}, \quad \overline{\mathcal{C}_j} \cap \overline{\mathcal{C}_k} \text{ has measure zero when } j \neq k \text{ and } \xi^0 \notin \overline{\mathcal{C}_j} \text{ for } j \geq 1.$$

Since $\xi^0 \notin \overline{\mathcal{C}_j}$ and $\mathcal{C}_j$ is acute, we can get acute, open cones $\Gamma^j,\ 1 \leq j \leq n$ and a constant $c > 0$ such that

$$\xi^0.\Gamma^j < 0 \text{ and } y.\xi \geq c|y||\xi|,\ \forall y \in \Gamma^j, \forall \xi \in \mathcal{C}_j.$$

We then have

$$u_0^\epsilon(z) = \sum_{j=1}^n \int_{\mathcal{C}_j} \int_{|x'| \leq a} e^{i\xi.(z-x') - \epsilon^2|\xi|^2} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

For $j = 0, 1, \ldots, n$, and $z = x + iy \in \mathbb{R}^m + i\Gamma^j$, define

$$f_j^\epsilon(x+iy) = \int_{\mathcal{C}_j} \int_{|x'| \leq a} e^{i\xi.(x+iy-x') - \epsilon^2|\xi|^2} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Then $f_j^\epsilon(z)$ are entire for $j \geq 1$ and converges uniformly on compact subsets of the wedge $\mathbb{R}^m + i\Gamma^j$ to the function

$$f_j(x+iy) = \int_{\mathcal{C}_j} \int_{|x'| \leq a} e^{i\xi.(x+iy-x')} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi,$$

which is holomorphic. Moreover, f_j increases M^*- exponentially on $\mathbb{R}^m + i\Gamma_\delta^j$ for some $0 < \delta \leq 1$. Indeed,

$$\begin{aligned}
|f_j(x+iy)| &= |\int_{\mathcal{C}_j}\int_{|x'|\leq a} e^{i\xi.(x+iy-x')}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi| \\
&\leq \int_{\mathcal{C}_j}\int_{|x'|\leq a} |e^{i\xi.(x+iy-x')}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}|dx'd\xi \\
&\leq \int_{\mathcal{C}_j}\int_{|x'|\leq a} e^{-\xi.y}|\mathcal{F}(\phi u)(x',\xi)||\xi|^{\frac{m}{2k}}dx'd\xi \\
&\leq C\int_{\mathcal{C}_j} e^{-c|y||\xi|}e^{M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \text{ from Theorem 3.2} \\
&= C\int_{\mathcal{C}_j} e^{[M(|L\xi|)-c|y||\xi|}|\xi|^{\frac{m}{2k}}d\xi.
\end{aligned}$$

Now since for each $\varrho > 0$ there exist $c_\varrho > 0$ such that (see Lemma 2.4, (2.10))

$$M(L|\xi|) = 2M(L|\xi|) - M(L|\xi|) \leq \varrho M(c_\varrho L|\xi|) - M(L|\xi|),$$

choosing $L = L_\varrho \doteq \frac{c}{c_\varrho}$, we have

$$\begin{aligned}
|f_j(x+iy)| &\leq C\int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L|\xi|)-c|y||\xi|}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= C\int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L_\varrho|\xi|)-L_\varrho c_\varrho|y||\xi|}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= C\int_{\mathcal{C}_j} e^{\varrho M(c_\varrho L_\varrho|\xi|)-\frac{L_\varrho c_\varrho}{\varrho}|y||\xi|]}e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&\leq Ce^{\varrho\sup_{r>0}[M(r)-r\frac{|y|}{\varrho}]}\int_{\mathcal{C}_j} e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&= Ce^{\varrho\omega^*(\frac{|y|}{\varrho})}\int_{\mathcal{C}_j} e^{-M(L|\xi|)}|\xi|^{\frac{m}{2k}}d\xi \\
&\leq C_1 e^{\varrho M^*(\frac{|y|}{\varrho})}.
\end{aligned}$$

Thus each f_j, $j = 1,\ldots,n$ has boundary value $bf_j \in D^{M'}(\mathbb{R}^m)$.
Let

$$g_0^\epsilon(x) = \int_\Gamma\int_{|x'|\leq a} e^{i\xi.(x-x')-\epsilon^2|\xi|^2}\mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}}dx'd\xi.$$

By the estimate for $\mathcal{F}(\phi u)(x',\xi)$ on the set $\{x' : |x'| \leq a\} \times \Gamma$, $g_0^\epsilon(x)$ are smooth for all $\epsilon > 0$. And since $e^{-M(t)}$ is decreasing, and

$$\begin{aligned}
|g_0(x) - g_0^\epsilon(x)| &= \int_\Gamma\int_{|x'|\leq a} |\mathcal{F}(\phi u)(x',\xi)||\xi|^{\frac{m}{2k}}|1-e^{-\epsilon^2|\xi|^2}|dx'd\xi \\
&\leq C\int_\Gamma |\xi|^{\frac{m}{2k}}e^{-M(b|\xi|)}|1-e^{-\epsilon^2|\xi|^2}|d\xi \to 0 \text{ as } \epsilon \to 0^+.
\end{aligned}$$

By the monotone convergence theorem, $g_0^\epsilon(x)$ converge uniformly on $\mathbb{R}^m$ to the function

$$g_0(x) = \int_\Gamma \int_{|x'|\le a} e^{i\xi.(x-x')} \mathcal{F}(\phi u)(x',\xi)|\xi|^{\frac{m}{2k}} dx' d\xi.$$

Clearly g_0 is smooth on $\mathbb{R}^m$. In fact, $g_0 \in \mathcal{E}^M$ since

$$\begin{aligned}
|\partial^\alpha g_0(x)| &\le \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|} |\mathcal{F}(\phi u)(x',\xi)| dx' d\xi \\
&\le C \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|} e^{-M(b|\xi|)} d\xi \\
&= \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|} e^{-\frac{1}{2}M(b|\xi|)} e^{-\frac{1}{2}M(b|\xi|)} d\xi \\
&\le C \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|} \frac{(M_{2|\alpha|})^{1/2}}{(b|\xi|)^{|\alpha|}} e^{-\frac{1}{2}M(b|\xi|)} d\xi \quad \text{since } e^{-M(t)} \le \frac{M_j}{t^j}\ \forall j \\
&\le C \int_\Gamma |\xi|^{\frac{m}{2k}+|\alpha|} \frac{\left(AH^{2|\alpha|} M_{|\alpha|}^2\right)^{1/2}}{(b|\xi|)^{|\alpha|}} e^{-\frac{1}{2}M(b|\xi|)} d\xi \text{ by (2.3)} \\
&= C'' \int_\Gamma |\xi|^{\frac{m}{2k}} M_{|\alpha|} \frac{H^{|\alpha|}}{b^{|\alpha|}} e^{-\frac{1}{2}M(b|\xi|)} d\xi \\
&= C'' \left(\frac{H}{b}\right)^{|\alpha|} M_{|\alpha|} \int_\Gamma |\xi|^{\frac{m}{2k}} e^{-\frac{1}{2}M(b|\xi|)} d\xi \\
&\le C^{|\alpha|+1} M_{|\alpha|} \text{ by Lemma 2.4, (2.12).}
\end{aligned}$$

Thus there is $f_0(x,y) \in \mathcal{E}^M(V \times \mathbb{R}^m)$ such that $f_0(x,0) = g_0(x)$ and

$$|\frac{\partial f_0}{\partial \overline{z}_j}(x,y)| \le C^{N+1} \frac{M_N}{N!} |y|^N.$$

Choose Γ_0 an open cone such that $\xi^0.\Gamma_0 < 0$. Thus we have found open cones $\Gamma_0, \Gamma_1, \ldots, \Gamma_n$ and functions $f_j, j \ge 1$ holomorphic on $\mathbb{R}^m + i\Gamma_j^\delta$ (for some $0 < \delta < 1$), which increases M^*- exponentially and $f_0(x,y)$ in $\mathcal{E}^M(\mathbb{R}^m + i\Gamma_0^\delta)$ (for some $0 < \delta < 1$) which also increases M^*- exponentially and such that

$$\xi^0.\Gamma_j < 0,\ 0 \le j \le n \ \text{ and}$$

$$|\frac{\partial f_j}{\partial \overline{z}_k}(x,y)| \le C^{N+1} \frac{M_N}{N!} |y|^N,\ \forall j = 1,\ldots,n,\ \forall k = 0,1,\ldots,m,\ \forall N \in \mathbb{N}.$$

In the sense of ultradistributions, for $j = 1,\ldots,n$, we have

$$\lim_{\Gamma_j \ni y \to 0} f_j(x+iy) = \lim_{\epsilon \to 0^+} f_j^\epsilon(x) \ \text{ and } \lim_{\Gamma_0 \ni y \to 0} f_0(x+iy) = \lim_{\epsilon \to 0^+} g_0^\epsilon(x).$$

Hence $u_0(x) = \sum_{j=0}^n b f_j$ in $D^{M'}(\mathbb{R}^m)$.
This implies that u_0 is $M-$ microregular at $(0,\xi^0)$. The proof is completed except for the claim that $u_1^\epsilon(z)$ converges to a holomorphic function $u_1(z)$, which follows from the lemma below, which can be proved by following the lines of proofs of Lemma 4.1 of Berhanu and Hounie (2012). In that case u_1 is microlocally analytic

at $(0,\xi^0)$ and so u_1 is $M-$microregular at $(0,\xi^0)$. Thus, we conclude that u is $M-$microregular at $(0,\xi^0)$. □

Lemma 3.5. *There is $\delta > 0$ and a holomorphic function $u_1(z) \in \mathcal{O}(B_\delta)$, $B_\delta \subset \mathbb{C}^m$ such that $\lim_{\epsilon\to 0^+} u_1^\epsilon(z) = u_1(z)$, $z \in B_\delta$.*

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